UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA Y TOPOLOGÍA

TRABAJO DE FIN DE MÁSTER

Monodromy on isolated hypersurface singularities

Tutora: María Pe Pereira

Inés García Redondo

Máster en Matemáticas Avanzadas

Curso académico 2020-21

Convocatoria ordinaria

Madrid, 1 de Julio 2021

Contents

1 Isolated hypersurface singularities **1**
- 1.1 Milnor's Fibration 1
  - 1.1.1 Transversality and good representatives of the germ . 2
  - 1.1.2 Milnor's Fibration 3
- 1.2 Morsifications of isolated singularities 5
- 1.3 Topology of the fibre of an isolated singularity 7
  - 1.3.1 Construction of a model with the homotopy type of a bouquet of spheres in the Milnor fibre 8
- 1.4 Milnor number of isolated singularities 11

2 Picard-Lefschetz Theory **15**
- 2.1 The monodromy and variation operators 15
- 2.2 The monodromy of the Morse singularity 18
  - 2.2.1 Particular cases of dimension $n = 0$ and $n = 1$ 18
  - 2.2.2 General case: Picard-Lefschetz theorem 20
- 2.3 Studying the monodromy from a morsification 23
- 2.4 Vanishing cycles and the Intersection Matrix 26

3 Direct sum of singularities **29**
- 3.1 Direct sum of singularities 29
- 3.2 Homology of the join of two topological spaces 30
- 3.3 The Variation Operator of the direct sum 31

4 The Cohomology Bundle **35**
- 4.1 The cohomology bundle 35
- 4.2 Connections on bundles, locally flat connections and holonomy 37
- 4.3 Horizontal sections and covariant derivatives 40
- 4.4 Application to isolated singularities 42
  - 4.4.1 Holomorphic structure 42
  - 4.4.2 Analytic triviality 43
  - 4.4.3 Horizontal sections 43

5 Computing the cohomology from the complex of forms **45**

5.1 Cohomology of sheaves 45
5.1.1 Čech Cohomology 45
5.1.2 Alternative definition of the cohomology 47
5.2 Alternative definition of the cohomology bundle with sheaves 50
5.3 Constant sheaves and simplicial cohomology 51
5.4 Abstract de Rham Theorem 53
5.5 Coherent sheaves 59
5.6 Stein manifolds 62
5.7 Computing the cohomology from the complex of holomorphic forms 65

6 Integrals of holomorphic forms over cycles in the fibres 68
6.1 Holomorphic dependence on parameters 68
6.2 Branches of the integrals 72
6.3 Expansion of the integral in series 73
6.4 General hypersurface singularities 76

Resumen

La monodromía estudia cómo cambian ciertos objetos matemáticos cuando "se mueven" en torno a una singularidad. En este trabajo, estaremos interesados en estudiar la monodromía en el caso particular de singularidades aisladas de hipersuperficies. Sea $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ el germen de una función analítica con un punto crítico aislado en el origen $\underline{0} \in \mathbb{C}^{n+1}$.

En el capítulo 1 se estudia la teoría básica relacionada con las singularidades aisladas de hipersuperficies. Revisamos el conocido Teorema de la fibración de Milnor. Este establece que para todo $\rho > 0$ suficientemente pequeño existe un $\epsilon > 0$ tales que la restricción

$$f|_{f^{-1}(\mathbb{D}_\epsilon^*)\cap\mathbb{B}_\rho} : f^{-1}(\mathbb{D}_\epsilon^*) \cap \mathbb{B}_\rho \to \mathbb{D}_\epsilon^*,$$

es localmente trivial, donde $\mathbb{D}_\epsilon$ es el disco cerrado de radio $\epsilon > 0$ centrado en $0 \in \mathbb{C}$ y $\mathbb{B}_\rho$ la bola cerrada de radio $\rho > 0$ centrada en el origen $\underline{0} \in \mathbb{C}^{n+1}$. La fibra de esta fibración se denomina fibra de Milnor. Tras esto, se estudia una familia de perturbaciones concretas de la función f llamada morsificación. A partir de esta, podremos demostrar que la fibra de Milnor tiene el tipo de homotopía de un bouquet de esferas de dimensión real n. Terminamos el capítulo demostrando que el número de esferas que aparecen en dicho bouquet es precisamente el número de Milnor de la singularidad.

En el capítulo 2 comenzamos el estudio de la monodromía de la singularidad propiamente dicha. En la primera sección definiremos los operadores de monodromía geométrica y algebraica y el operador de variación. En la segunda sección aplicamos estos conceptos a un ejemplo de particular interés: el de la singularidad de Morse. Usaremos los resultados ahí obtenidos en la tercera sección para estudiar más a fondo la monodromía de una singularidad aislada de hipersuperficie general utilizando una morsificación del germen que la define. Concluimos el capítulo introduciendo los ciclos evanescentes y la matriz de intersección de la singularidad.

En el capítulo 3 estudiamos lo anterior aplicado a un ejemplo concreto de relevancia en el campo de la teoría de singularidades: la suma directa de singularidades.

En el capítulo 4 utilizaremos la teoría de conexiones definidas sobre fibrados para estudiar la monodromía algebraica. En la primera sección del capítulo intoducimos el fibrado de cohomología, un fibrado vectorial complejo que se puede construir a partir de cualquier fibrado diferenciable. En particular, lo podemos obtener para la fibración de Milnor. En la segunda sección introducimos la teoría de conexiones definidas sobre fibrados, y la utilizamos para reconocer una conexión particular en el fibrado de cohomología. Esta será una conexión lineal y localmente plana. Su holonomía nos dará precisamente la monodromía algebraica de la singularidad. Esta holonomía se concreta en las secciones planas de la conexión. Este tipo de secciones se caracterizan en la sección cuarta del capítulo por medio de las derivadas covariantes. Finalmente, aplicamos todo lo anterior al caso de la fibración de Milnor. El hecho de que la conexión sea localmente plana y que estemos trabajando con un fibrado complejo sobre una variedad compleja nos permitirá dotar al fibrado de cohomología de una estructura holomorfa.

En el capítulo 5 se justifica la aparición de formas holomorfas en el estudio clásico de la monodromía de singularidades. El objetivo del capítulo será encontrar un isomorfismo entre la cohomología del complejo de formas holomorfas definidas en la fibra de Milnor y la cohomología singular de dicha fibra. Este isomorfismo se inspira en el que se tiene por el Teorema de de Rham para formas diferenciables sobre variedades diferenciables reales. Para lograrlo se utiliza la teoría de cohomología de haces sobre espacios complejos. De esta manera, este capítulo es una exposición de los contenidos relacionados con esta teoría necesarios para encontrar el isomorfismo que nos interesa: haces constantes y cohomología simplicial, el teorema abstracto de de Rham, haces coherentes, variedades Stein y los teoremas de Cartan A y B. El isomorfismo obtenido estará dado por las integrales de las formas holomorfas sobre cadenas diferenciables en la fibra de Milnor.

Motivados por este resultado, en el capítulo 6 estudiamos este tipo de integrales. Demostraremos que definen funciones holomorfas multi-evaluadas definidas en el disco punteado $\mathbb{D}_\epsilon \setminus \{0\}$, base de la fibración de Milnor. Es más: veremos que en el origen $0 \in \mathbb{C}$ admiten un desarrollo en serie con unas determinadas propiedades, relacionadas con la monodromía algebraica. Con esto volvemos al objeto de interés del trabajo y concluimos la exposión.

Abstract

Roughly speaking, the monodromy is the study of how some mathematical objects change when they "go round" a singularity. In this work, we will be interested in studying this situation for isolated hypersurface singularities. Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at the origin $\underline{0} \in \mathbb{C}^{n+1}$.

In Chapter 1 we study the basic theory related to isolated hypersurface singularities. We review the famous Milnor's Fibration Theorem, which states that for every $\rho > 0$ sufficiently small there exists some $\epsilon > 0$ such that the restriction

$$f|_{f^{-1}(\mathbb{D}_\epsilon^*)\cap\mathbb{B}_\rho} : f^{-1}(\mathbb{D}_\epsilon^*) \cap \mathbb{B}_\rho \to \mathbb{D}_\epsilon,$$

is a smooth fibre bundle, where $\mathbb{D}_\epsilon$ is the closed disk of radius $\epsilon > 0$ centred at $0 \in \mathbb{C}$ and $\mathbb{B}_\rho$ the closed ball of radius $\rho > 0$ centred in $\underline{0} \in \mathbb{C}^{n+1}$. The fibre of this fibration is called Milnor fibre. Afterwards, we study a particular family of deformations of the previous germ, called the morsification the germ. From this family we will be able to prove that the fibres from the previous bundle have to homotopy type of a bouquet of spheres of real dimension n. We end this chapter showing that the number of spheres appearing in that bouquet is precisely the Milnor number of the singularity.

In chapter 2 we begin to properly study the monodromy around the singularity defined by f. In the first section, we explicitly construct the geometric and algebraic monodromy operators and the variation operator of an isolated hypersurface singularity. In the second section we apply these concepts to a very important example: the Morse singularity. This will be used in section 3 to study the monodromy of a general isolated hypersurface singularity from a morsification of the germ defining the singularity. We end the chapter by defining the vanishing cycles and the Intersection Matrix of the singularity.

In chapter 3 we introduce an important example in singularity theory: the direct sum of singularities. We study the monodromy and variation operators applied to this case.

In chapter 4 we use the theory of connections on bundles to study the alge-

braic monodromy. In the first section we define the cohomology bundle, a complex vector bundle associated to any smooth fibre bundle. In particular, it can be constructed for the Milnor Fibration. Afterwards, we review the theory of connections on bundles and use it to recognise a particular connection on the cohomology bundle. This connection will turn out to be linear and locally flat. Its holonomy will give us the algebraic monodromy of the singularity. This holonomy can be realised by the horizontal sections. We characterise those in the third section of the chapter with covariant derivatives. We end the chapter applying all the previous contents to isolated hypersurface singularities. From the locally flatness of the connection and since we are working with a complex vector bundle over a complex manifold, we will be able to define a holomorphic structure for the cohomology bundle and the connection of isolated hypersurface singularities.

In chapter 5 we justify the appearance of forms in the classical study of the monodromy singularities. The objective there will be to find an isomorphism between the cohomology obtained from the complex of holomorphic forms over the Milnor fibres and its singular cohomology with coefficients in $\mathbb{C}$. This isomorphism will be based in the de Rham theorem known for differential forms over real manifolds. To achieve this goal, we need to work with cohomology of sheaves over complex spaces. Therefore, this chapter is an exposition the contents related to this theory which are necessary to obtain the desired isomorphism: constant sheaves and simplicial cohomology, the abstract de Rham theorem, coherent sheaves, Stein manifolds and Cartan's theorems A and B. The isomorphism of our interest will be defined by the integral of holomorphic forms over smooth chains on the complex manifold.

Motivated by this last result, in chapter 6 we study those kind of integrals. We will show that they define multi-valuate holomorphic functions defined in the punctured target disk $\mathbb{D}_\epsilon \setminus \{0\}$ of the Milnor's Fibration. What is more: we will prove that in the origin $0 \in \mathbb{C}$ they admit a series expansion with certain properties, closely related to the algebraic monodromy.

Chapter 1

Isolated hypersurface singularities

We begin this work with an introductory chapter in which we will state some of the fundamental definitions and results that we will need afterwards, and fix the notation that we will be using in the following chapters. Part of the theory that here will be explained was covered in the final degree thesis that I developed last year, also supervised by María Pe Pereira, following the classical references on the topic by Milnor [1] and Arnold [3]. In that work you shall find all the details that we will omit here.

We will study here many different properties and geometric information about a germ $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ of an analytic function with a critical point at the origin.

1.1 Milnor's Fibration

We want to study the local properties of isolated singularities of complex hypersurfaces, which are described locally as the zero level set of an analytic function. To simplify our notations, we will suppose that the singular point which we are studying is at the origin $\underline{0} \in \mathbb{C}^{n+1}$. We begin introducing the objects that we will use to describe that situation. We suppose that the reader is familiar with the notion of germ of a space and germ of a function. If that is not the case, a nice introduction to this theory shall be found in the third chapter of [2].

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of analytic function at the origin $\underline{0} \in \mathbb{C}^{n+1}$ which verifies $f(\underline{0}) = 0$. We can consider the germ of the set $X := f^{-1}(0)$ at the origin $\underline{0} \in \mathbb{C}^{n+1}$, which we denote by $(X, \underline{0})$. This defines a germ of hypersurface at the origin. We will always consider that the germ $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$, from which we parted, is a generator of the ideal $\mathcal{I}(X, \underline{0}) \subset \mathcal{O}_{n+1}$.

The Implicit Function Theorem tells us that the level sets $f^{-1}(\epsilon)$ are manifolds when ϵ is not a critical value of f. Moreover, if, for example $\epsilon = 0$ is a critical value and its associated critical point is isolated and located at the origin, the level set $f^{-1}(0)$ is also a manifold away from the origin.

Additionally, the **Jacobian Criterion** allows us to state the following.

Theorem 1.1.1. *Let $(X, \underline{0})$ be the germ of an analytic hypersurface and $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function generating the ideal $\mathcal{I}(X, \underline{0})$. Then, the following conditions are equivalent.*

1. *The germ of hypersurface $(X, \underline{0})$ is not smooth.*
2. *The analytic function $f \in \mathcal{O}_{n+1}$ has a critical point at the origin.*

Furthermore, saying that $(X, \underline{0}) = (f^{-1}(0), \underline{0})$ has an isolated singularity is equivalent to saying that f has an isolated critical point at the origin $\underline{0} \in \mathbb{C}^{n+1}$.

In 1968, John Milnor proved several important facts concerning singularities in complex hypersurfaces. Before stating them, we make some observations about the representatives of the germs we will be taking.

1.1.1 Transversality and good representatives of the germ

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be an analytic function with an isolated critical point at the origin. The following results were proved by Milnor in [1], and were covered in the second chapter of my final degree thesis.

Lemma 1.1.1. *There exists $\rho > 0$ sufficiently small such that for every $0 < r \leq \rho$, the sphere $\mathbb{S}_r$ of radius r centred at the origin $\underline{0} \in \mathbb{C}^{n+1}$ intersects transversely the level set $f^{-1}(0) = V(f)$.*

With the previous lemma in mind, Milnor proved the **Conic Structure Theorem**, which states that for that $\rho > 0$ sufficiently small, the intersection of the closed ball centred at the origin with radius $\rho > 0$ with the hypersurface: $\mathbb{B}_\rho \cap V(f)$, is homeomorphic to the cone of the space $\mathbb{S}_\rho \cap V(f)$ over the origin, that is

$$C(\mathbb{S}_\rho \cap V(f)) := \{tx : \ t \in [0, 1], \ x \in \mathbb{S}_\rho \cap V(f)\}.$$

The value of $\rho > 0$ verifying lemma 1.1.1 is commonly known as **Milnor radius**, and the closed ball $(\mathbb{B}_\rho)$ and sphere $(\mathbb{S}_\rho)$ of that radius are called **Milnor ball** and **Milnor sphere**. The intersection $\text{Link}(f, \underline{0}) := \mathbb{S}_\rho \cap V(f)$ is called the **link of the singularity**. Its topology does not depend on the Milnor radius.

In particular, from the Conic Structure theorem we get that the representative of X in a Milnor ball is contractible.

Since being transverse is an open condition, we also have the following lemma.

Lemma 1.1.2. *For sufficiently small $\epsilon > 0$ the target disk $\mathbb{D}_\epsilon$ only contains one critical value of f, namely, the $0 \in \mathbb{C}$, and the level sets $f^{-1}(z)$ for $z \in \mathbb{D}_\epsilon := \{z \in \mathbb{C} : |z| \leq \epsilon\}$ also intersect transversely the Milnor sphere $\mathbb{S}_\rho$.*

With all these results in mind, we fix the following notations for the remaining of this text:

- $D := \mathbb{D}_\epsilon$ and $D^* := \mathbb{D}_\epsilon \setminus \{0\}$,
- $X := \mathbb{B}_\rho \cap f^{-1}(D)$,
- for $z \in D$ the fibre $X_z := f^{-1}(z) \cap \mathbb{B}_\rho$,
- $X^* := X \setminus X_0$.

In this work, unless otherwise stated, when we take a representative of the germ defining the hypersurface, we will consider it defined between the following spaces

$$f : X \to D.$$

We will call this a **good representative** of the germ.

1.1.2 Milnor's Fibration

Now, we give an overview of one of the most important theorems introduced by Milnor in [1]: his famous **Milnor's Fibration Theorem**. The theory of this section was covered in the third chapter of my final degree thesis. We will state here three different versions of the Milnor's Fibration, being the third the one we will use most in this work. These theorems can be proved using the Ehresmann theorem, which we include next since we will refer to it in several occasions.

Theorem 1.1.2 (Ehresmann's Theorem)**.** *Let M and N be two smooth manifolds without boundary and let $f : M \to N$ be a smooth surjective map which is a proper map and a submersion. Then, f is locally trivial.*

Moreover, if $\partial M \neq \emptyset$ and additionally we have that $f|_{\partial M}$ is a submersion, then we can still conclude that f is locally trivial.

We begin with the version which Milnor introduced first.

Theorem 1.1.3 (**Milnor's Fibration on the Milnor sphere**)**.** *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of analytic function with an isolated critical* *point at the origin, and $\rho > 0$ its Milnor radius. Then, the argument function*

$$\phi := \frac{f}{|f|} : \mathbb{S}_\rho \setminus \mathrm{Link}(f, \underline{0}) \to \mathbb{S}^1$$

describes a smooth fibre bundle.

This means that for every $e^{i\theta} \in \mathbb{S}^1$, there exists a neighbourhood $U \subset \mathbb{S}^1$ of $e^{i\theta}$, a smooth manifold F and a diffeomorphism $\varphi : \phi^{-1}(U) \to U \times F$ such that the following diagram is commutative

$$\begin{array}{ccc} \phi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ {\scriptstyle \phi}\big\downarrow & \swarrow_{\pi_1} & \\ U & & \end{array}$$

where $\pi_1 : U \times F \to U$ is the first projection.

There is another version of this theorem, which describes a smooth fibre bundle equivalent to the previous one. It relies heavily on the transversality conditions of the previous section.

Theorem 1.1.4 (**Milnor's Fibration on the Milnor tube**)**.** *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of analytic function with an isolated critical point at the origin, $\rho > 0$ its Milnor radius and $\epsilon > 0$ in the conditions of lemma 1.1.2. Then, f is locally trivial when restricted to $f^{-1}(\partial\mathbb{D}_\epsilon) \cap \mathbb{B}_\rho$. That is, we have the following smooth fibre bundle*

$$f|_{f^{-1}(\partial\mathbb{D}_\epsilon)\cap\mathbb{B}_\rho} : f^{-1}(\partial\mathbb{D}_\epsilon) \cap \mathbb{B}_\rho \to \partial\mathbb{D}_\epsilon.$$

*The space $f^{-1}(\partial\mathbb{D}_\epsilon) \cap \mathbb{B}_\rho$ is usually referred to as the **Milnor tube**.*

A consequence of the previous fibrations being equivalent is that their fibres are diffeomorphic. That is, we have that

$$\phi^{-1}(e^{i\theta}) \cong f^{-1}(\delta) \cap \mathring{\mathbb{B}}_\rho$$

for every $e^{i\theta} \in \mathbb{S}^1$ and $\delta \in \partial\mathbb{D}_\epsilon$. The first term is a fibre of Milnor's Fibration on the sphere and the second term, an open fibre on the tube (observe that we intersected with the open ball). Both of them, since they are diffeomorphic, are referred to as **Milnor fibres**.

Finally, we can consider a third fibration which we will use a lot in this work. This fibration is obtained considering all the possible tubes $f^{-1}(\partial\mathbb{D}_r) \cap \mathbb{B}_\rho$, for $0 < r \leq \epsilon$, which are in the conditions of transversality of lemma 1.1.2 as well.

Theorem 1.1.5 (**Milnor's Fibration on the filled tube**)**.** *Consider $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ a germ of analytic function with an isolated critical point* *at the origin, $\rho > 0$ its Milnor radius and $\epsilon > 0$ in the conditions of lemma 1.1.2. Then, f is locally trivial when restricted to $f^{-1}(\mathbb{D}^*_\epsilon) \cap \mathbb{B}_\rho$. That is, the restriction of a good representative of the germ*

$$\pi := f|_{X^*} : X^* \to D^*$$

is locally trivial.

Its fibres are again the Milnor fibres.

1.2 Morsifications of isolated singularities

Now, let us study non-degenerate singularities, which give rise to Morse functions. We will see that, in this case, we know the diffeomorphic type of the fibres of the Milnor's fibration. After that, we will see how to deform a general analytic function with an isolated critical point into a family of Morse functions with some other useful characteristics. This theory is part of the last chapter of the final degree thesis aforementioned.

Let $f : U \to \mathbb{C}$ be an analytic function where $U \subset \mathbb{C}^n$. We recall that the Hessian of f in $x_0 \in U$ is the matrix with the second partial derivatives of that function

$$H_f(x_0) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right), \qquad 1 \leq i, j \leq n.$$

Definition 1.2.1. We say that f has a **non-degenerate critical point** in $x_0 \in U$ when it has a critical point in x_0 such that $H_f(x_0)$ is a non-degenerate matrix. These kind of critical points are also called **Morse points** or **Morse singularities**.

Definition 1.2.2. When we have a function which only has non-degenerate critical points, we call it a **Morse function**.

A very important feature of this special kind of functions is the following.

Lemma 1.2.1 (Morse's lemma)**.** *Let $f : U \to \mathbb{C}$ be an analytic function with $U \subset \mathbb{C}^n$ and having a non-degenerate critical point at $x_0 \in U$. Then there exists a local system of coordinates $(x_1, ..., x_n)$ defined in a neighbourhood $V \subset U$ of x_0 such that the function f has the following expresion in V:*

$$f(x_1, ..., x_n) = f(x_0) + \sum_{i=1}^{n} x_j^2.$$

In the framework here presented, we will say that the germ of analytic function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ defines a **Morse singularity** when it has a representative which is Morse, that is, which has a non-degenerate critical point at the origin. We can assume that this representative is a good representative $f : X \to D$. In this situation, we have the diffeomorphic type of the Milnor fibres completely determined due to the following lemma.

Lemma 1.2.2. *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated Morse singularity at the origin, $\mathbb{B}_\rho$ its Milnor ball and $\mathbb{D}_\epsilon$ in the conditions of lemma 1.1.2. Then $X_z := f^{-1}(z) \cap \mathbb{B}_\rho$ for $z \in \mathbb{D}^*_\epsilon$ is diffeomorphic to the disk subbundle of the tangent bundle of the standard n-dimensional sphere $\mathbb{S}^n$.*

Proof. Let us first make some assumptions that will simplify the problem. First, we know by the Morse Lemma that in a neighbourhood of the origin, which is a non-degenerate critical point, we can take a system of coordinates such that

$$f(x_1, ..., x_{n+1}) = \sum_{i=1}^{n+1} x_j^2.$$

After a linear change of coordinates, we can assume that $\epsilon = 1$. Since the Milnor fibres are all diffeomorphic, we only need to prove the lemma for the fibre X_1, for instance.

If we consider the real and imaginary part of each coordinate $x_j = u_j + iv_j$, where u_j and v_j are real, we have the following equations describing X_1:

$$\sum_{j=1}^{n+1} u_j^2 - \sum_{j=1}^{n+1} v_j^2 = 1, \qquad \sum_{j=1}^{n+1} u_j v_j = 0, \qquad \sum_{j=1}^{n+1} u_j^2 + \sum_{j=1}^{n+1} v_j^2 \leq \rho^2.$$

Taking the transformation

$$\tilde{u}_j = \frac{u_j}{\sqrt{\sum u_j^2}}, \qquad \tilde{v}_j = v_j$$

we get the equations

$$\sum_{j=1}^{n+1} \tilde{u}_j = 1, \quad \sum_{j=1}^{n+1} \tilde{u}_j \tilde{v}_j = 0, \quad \sum_{j=1}^{n+1} \tilde{v}_j \leq \frac{\rho^2 - 1}{2}$$

which are the ones describing a disk subbundle of the tangent bundle of the standard n-dimensional sphere in $\mathbb{R}^{2n-1}$ defined by

$$\mathbb{S}^n = \{(x_1, ..., x_{n+1}) : \sum_{j=1}^{n+1} u_j^2 = 1, v_j = 0\}$$

as we wanted. □

From the previous lemma we can see that we have a great deal of information about the particular case of Morse singularities but, quite obviously, not every germ $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ which defines a singularity is Morse. However, we will be able to find a perturbation of $f : X \to D$ a good representative of the germ, sufficiently close to it with some very convenient properties, among them, being Morse.

Theorem 1.2.1. *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ with an isolated critical point at the origin and $\mathbb{B}_\rho$ its Milnor ball. We can construct a family*

$$\{f_\lambda : \lambda \in \mathbb{C}, |\lambda| \leq \lambda_0\}$$

of perturbations $f_\lambda : X_\lambda \to D$, where $X_\lambda := f_\lambda^{-1}(D) \cap \mathbb{B}_\rho$, verifying that:

- *$f_0 = f$,*
- *the fibres $f_\lambda^{-1}(s)$ are transverse to the sphere $\mathbb{S}_\rho$ for every $|\lambda| \leq \lambda_0$ and $s \in D$,*
- *f_λ only has a finite number of critical points, all non-degenerate and lying in $\mathring{D}$,*
- *and that the critical values associated to those critical points are all different from each other.*

This is what we will call a **morsification** of the singularity. The number of critical values that we obtain under a perturbation of this kind is always the same, and equals the Milnor number of the singularity $\mu(f)$. We will give an idea of why this happens in the following two sections of this chapter.

Lastly, if we fix some $0 < |\lambda| \leq \lambda_0$, maybe taking a smaller λ_0, we can ensure that the Milnor fibres of the function f

$$X_z := f^{-1}(z) \cap \mathbb{B}_\rho, \qquad \text{for } 0 < |z| \leq \epsilon.$$

are diffeomorphic to the fibres of the function from the morsification

$$X_{z,\lambda} := f_\lambda^{-1}(z) \cap \mathbb{B}_\rho \qquad \text{for } 0 < |z| \leq \epsilon.$$

1.3 Topology of the fibre of an isolated singularity

We finished the final degree thesis calculating the homotopy type of the Milnor fibres in the case of isolated singularities, following the approach of Arnold in [3]. We devote this section to stating these last results. Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at the origin.

Note that we already know the homotopy type of the Milnor fibres for isolated **Morse singularities**. Thanks to lemma 1.2.2, we can assure that they have the homotopy type of a sphere of real dimension n.

Let us go back to the general case and recall the notations of section 1.1.1. We know that the fibres

$$X_z := f^{-1}(z) \cap \mathbb{B}_\rho, \qquad \text{for } 0 < |z| \leq \epsilon.$$

are compact manifolds of complex dimension n with boundary $\partial X_z = X_z \cap \mathbb{S}_\rho$. Moreover, since we have the structure of fibre bundle described in theorem 1.1.5, we can assure that they are all diffeomorphic.

In theorem 6.5 of [1] Milnor proved that the manifold X_z has the homotopy type of a bouquet of spheres of real dimension n, and then in the following chapter he showed that the number of spheres of that bouquet coincides with the Milnor number of the singularity. In [3], Arnold, following Brieskorn [17], gave another approach to this using morsifications and the fact that we know the homotopy type of Morse singularities. Next, we explain briefly this second method, since it entails a construction which has some interest for our purposes. The situation which we describe next is pictured in the figure 1.1.

1.3.1 Construction of a model with the homotopy type of a bouquet of spheres in the Milnor fibre

Consider $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ a germ of analytic function with an isolated singularity at the origin, and let $\mathbb{B}_\rho$ be its Milnor ball and $\mathbb{D}_\epsilon$ the disk in the conditions of lemma 1.1.2. We take a morsification $\{f_\lambda : \lambda \in \mathbb{C}, |\lambda| \leq \lambda_0\}$ of that germ, that is, a deformation of f in the conditions of theorem 1.2.1. Let us fix some $|\lambda| \leq \lambda_0$ and its respective function from the morsification f_λ. We will construct a model with the homotopy type of the desired bouquet of spheres which is contained in a fibre $X_{s,\lambda}$ of f_λ.

We know that f_λ has a finite number of critical points p_i, for $i = 1, ..., N$, which are all non-degenerate and whose critical values $z_i = f_\lambda(p_i)$ are pairwise distinct: $z_i \neq z_j$ for $i \neq j$. Let $\mathbb{B}_{\rho_i} \subset \mathbb{B}_\rho$ be the Milnor ball of the critical point p_i and $\mathbb{D}_{\epsilon_i} \subset \mathbb{D}_\epsilon$ be the disks in the conditions of lemma 1.1.2 for those points as well. Since the critical points p_i are non-degenerate, that is, Morse singularities, we know that the space $f_\lambda^{-1}(s_i) \cap \mathbb{B}_{\rho_i}$ for some $s_i \in \partial\mathbb{D}_{\epsilon_i}$ has the homotopy type of a n-dimensional sphere.

Now, we take $z_0 \in \partial\mathbb{D}_\epsilon$ and a set of disjoint paths $u_i : I \to D$ joining s with the critical values z_i, intersecting only once the disks $\mathbb{D}_{\epsilon_i}$ in a set of points $s_i \in \partial\mathbb{D}_{\epsilon_i}$, and not passing through any other of the critical values. We call α_i to the first part of those paths, that is, the one joining z_0 with the points s_i. We define

$$\Gamma := \bigcup_{i=1}^{N} \alpha_i.$$

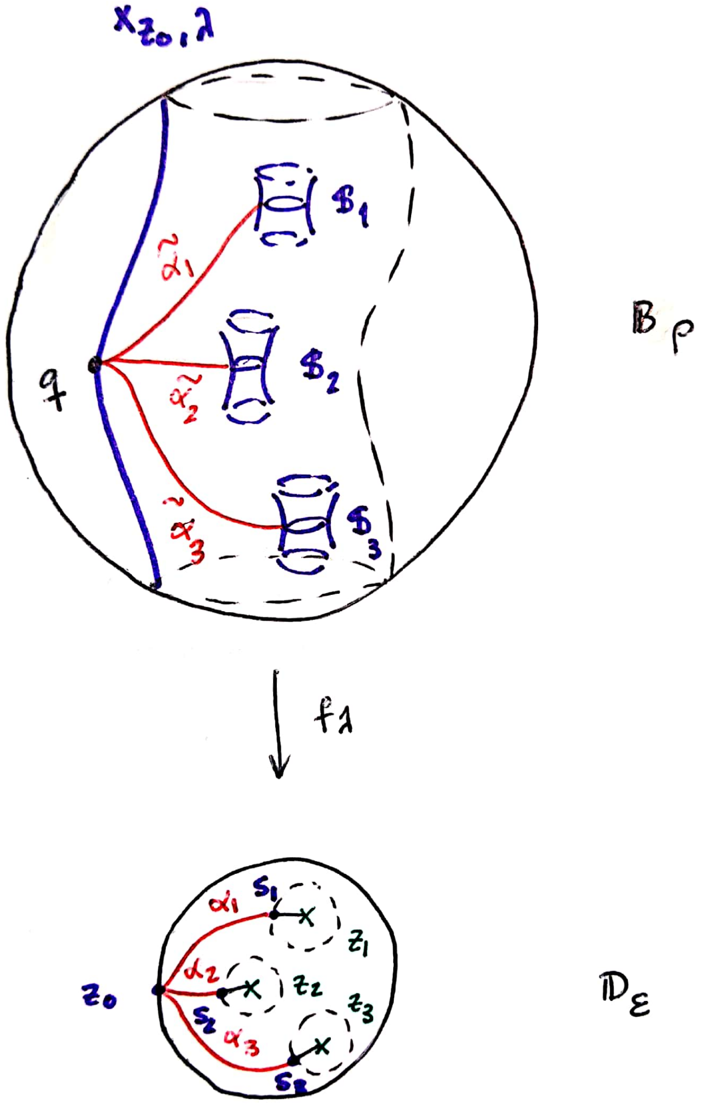

Figure 1.1: Model with the homotopy of a bouquet of n-dimensional spheres in the Milnor fibre of a morsification

We fix some $q \in X_{z_0,\lambda}$ and lift the paths α_i to some $\tilde{\alpha_i}$ verifying $\alpha_i = f_\lambda \circ \tilde{\alpha_i}$ and which all start in q and end in some point over the fibre $f_\lambda^{-1}(s_i) \cap \mathbb{B}_{\rho_i}$.

Now, let us build spaces $\mathbb{S}_i \subset f_\lambda^{-1}(s_i) \cap \mathbb{B}_{\rho_i}$, for $i = 1, ..., N$, with the homotopy type of a n-dimensional sphere. By the Morse Lemma, we know that in a neighbourhood of the critical point p_i there is a system of coordinates $(x_1, ..., x_{n+1})$ such that the function f_λ can be written as

$$f_\lambda(x_1, ..., x_{n+1}) = z_i + \sum_{i=1}^{n+1} x_i^2.$$

Thus, for values of the parameter $t > 0$ of the path u_i sufficiently close to zero we can fix in the level manifold $X_{u_i(t),\lambda}$ the following sphere

$$\mathbb{S}(t) := \sqrt{u_i(t) - z_i}\ \mathbb{S}^n$$

where

$$\mathbb{S}^n := \{(x_1, ..., x_{n+1}) : \quad \sum_{i=1}^{n+1} x_i^2 = 1; \quad \mathrm{Im}(x_i) = 0, \quad \forall i = 1, ..., n+1\}$$

is the standard unit n-dimensional sphere. Taking the value of the parameter $\tau_i \in I$ corresponding to the point $s_i = u_i(\tau_i)$, we obtain the sphere $\mathbb{S}_i = \mathbb{S}(\tau_i)$ we where looking for.

We then define

$$Y := \bigcup_{i=1}^{N} (\tilde{\alpha_i} \cup \mathbb{S}_i)$$

which has, by definition, the homotopy type of a bouquet of N spheres of real dimension n.

The space Y is contained in $f_\lambda^{-1}(\Gamma)$. We know that f_λ is locally trivial over Γ, and since Γ is a contractible set, we can conclude that f_λ is trivial over that space. Therefore, we have that the following spaces are diffeomorphic

$$f_\lambda^{-1}(\Gamma) \cong \Gamma \times X_{s,\lambda}.$$

Now, using again that Γ is contractible, we conclude that $f_\lambda^{-1}(\Gamma)$ has the same homotopy type than $X_{s,\lambda}$.

In conclusion, we have described a space Y contained in a space with the homotopy type of the fibre $X_{s,\lambda}$. One can prove, using homological arguments, that the previous inclusion is actually an homotopic equivalence (see theorem 2.2 in [3]). Thus, the fibre $X_{s,\lambda}$ has the homotopy type of a bouquet of N spheres of real dimension n. Since we can choose the morsification so the fibres $X_{s,\lambda}$ are diffeomorphic to the Milnor fibres of f, we arrive to the conclusion that we wanted: the fibre X_z has also the homotopy type of a bouquet of N spheres of dimension n. The homology groups of X_z are thus

$$H_k(X_z) = 0, \quad \text{for} \quad k \neq n; \qquad H_n(X_z) = \mathbb{Z}^N,$$

that is, the free abelian group with N generators.

Lastly, we realise that the number of spheres appearing in the bouquet is determined by the number of different Morse points that appear in a function from the morsification, which we denoted by N. Since the homotopy type of the fibre X_z only depends on the function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$, we conclude that this number of Morse points only depends on f as well, that is, $N = N(f)$. In the following section we will show that this number coincides with the Milnor number of the singularity, as Milnor had already proved.

1.4 Milnor number of isolated singularities

The objective here is proving that the number of Morse points appearing in the construction of the previous section equals the Milnor number of the fuction defining the singularity. We begin by defining it and giving a characterisation of isolated singularities in terms of the jacobian ideal and this Milnor number. Afterwards, we will introduce a key lemma for the purpose of the section: the Principle of Conservation of Number (applied to the Milnor number), and use it to prove it what we want to.

Let $f : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}, 0)$ be a germ of analytic function and

$$J(f)_p = \left(\left(\frac{\partial f}{\partial z_1} \right)_p, ..., \left(\frac{\partial f}{\partial z_{n+1}} \right)_p \right) \mathcal{O}_{n+1,p}$$

the jacobian ideal of f at p.

Definition 1.4.1. The **Milnor number** of f at p is defined as the following dimension

$$\mu(f, p) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,p}}{J(f)_p}.$$

The easiest example of the Milnor number of a point is that of a smooth point. In that case, the previous quotient is empty and we have that $\mu = 0$. For a **Morse singularity**, we can also easily compute the Milnor number. If we have $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ with a non-degenerate isolated critical point at the origin, we know that there exists a neighbourhood U of the origin and a system of coordinates $(x_1, ..., x_{n+1})$ such that the function f is of the form

$$f(x_1, ..., x_{n+1}) = x_1^2 + ... + x_{n+1}^2.$$

Therefore, we have that $J(f) = (x_1, ..., x_{n+1})\mathcal{O}_{n+1} = \mathfrak{m}_{n+1}$, and then, $\mu(f, \underline{0}) = 1$.

Now, let $\mathfrak{m}_{n+1} \subset \mathcal{O}_{n+1}$ be the maximal ideal of the ring of convergent series at the origin. Then, a necessary condition for $f \in \mathcal{O}_{n+1}$ to define a singularity at the origin is verifying $f \in \mathfrak{m}_{n+1}^2$. Moreover, the following lemma, which we will use in the future, can be proved.

Lemma 1.4.1. *Let $f \in \mathfrak{m}_{n+1}^2$. The following assertions are equivalent:*

1. *f has an isolated critical point at the origin,*
2. *$\sqrt{J(f)} = m_{n+1}$.*

We also have another characterisation of isolated singularities, concerning its Milnor number, for which we will include the proof.

Lemma 1.4.2. *Let $f \in \mathfrak{m}_{n+1}^2$. The following assertions are equivalent:*

1. *f has an isolated critical point at the origin,*
2. *$\mu(f, \underline{0}) < +\infty$.*

Proof. $\boxed{1. \Rightarrow 2.}$

If f has an isolated critical point at the origin, from lemma 1.4.1 and since $\mathcal{O}_{n+1}$ is noetherian, we know that there exists some $k \geq 0$ such that $\mathfrak{m}_{n+1}^k \subset J(f)$. Therefore we have a surjective homomorphism of $\mathbb{C}$-modules

$$\frac{\mathcal{O}_{n+1}}{\mathfrak{m}_{n+1}^k} \to \frac{\mathcal{O}_{n+1}}{J(f)}$$

which implies that the dimensions of those vector spaces verify

$$\dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n+1}}{J(f)} \right) \leq \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n+1}}{\mathfrak{m}_{n+1}^k} \right).$$

Since the space from the right is the vector space of polynomials of degree less than k, we get that its dimension as finite, and thus, that

$$\mu(f, \underline{0}) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n+1}}{J(f)} \right) < +\infty$$

as we wanted.

$\boxed{2. \Rightarrow 1.}$ We can consider the following chain of inclusions of ideals of $\mathcal{O}_{n+1}$:

$$J(f) \subset ... \subset \mathfrak{m}_{n+1}^k + J(f) \subset ... \subset m_{n+1}^2 + J(f) \subset m_{n+1} + J(f).$$

Since

$$\dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n+1}}{J(f)} \right) < +\infty$$

and in each of the previous inclusions we have

$$\dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{n+1}}{J(f)+\mathfrak{m}_{n+1}^{k+1}}\right) \geq \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{n+1}}{J(f)+\mathfrak{m}_{n+1}^{k}}\right)$$

we cannot have the strict inequality in an infinite number of inclusions. Therefore, the previous chain stabilises, that is, for some $k \geq 0$ we have that

$$\mathfrak{m}_{n+1}^{l} + J(f) = \mathfrak{m}_{n+1}^{l+1} + J(f)$$

for every $l \geq k$.

This fact implies the following equality

$$J(f) + \mathfrak{m}_{n+1}(J(f) + \mathfrak{m}_{n+1}^{k}) = J(f) + \mathfrak{m}_{n+1}^{k+1} = J(f) + \mathfrak{m}_{n+1}^{k}.$$

By Nakayama's Lemma we conclude that

$$J(f) = J(f) + \mathfrak{m}_{n+1}^{k}$$

that is,

$$\mathfrak{m}_{n+1}^{k} \subset J(f) \implies \sqrt{J(f)} = \mathfrak{m}_{n+1}$$

and again using the previous characterisation, we conclude that f has an isolated critical point at the origin. □

Now, we introduce an application of a very important theorem: the Principle of Conservation of Number. The motivation for that theorem is that some invariants related to singularities are characterised by the dimension of some vectorial spaces. That is exactly what happens with the Milnor number. The Principle of Conservation says that, under some reasonable hypothesis of coherence, a notion which will be introduce in the chapters to come, those dimensions are preserved under deformations of the germ f. We state an application of that theorem to the situation we are interested in.

Theorem 1.4.1. *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point, and $\mu(f, \underline{0})$ its Milnor number. Consider a deformation of f, that is*

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}, (\underline{0}, 0)) \to (\mathbb{C}, 0), \qquad F(x_1, ..., x_n, 0) = f(x_1, ..., x_n).$$

Then, for all sufficiently small open neighbourhoods U of $\underline{0} \in \mathbb{C}^{n+1}$ there exists an open neighbourhood V of $0 \in \mathbb{C}$ such that for all $s \in V$

$$\mu(f, \underline{0}) = \sum_{p \in U \times \{s\}} \mu(f_s, p)$$

where $f_s(x) := F(x, s)$.

The proof for this theorem might be checked on section 6.4 of [2]. With this fact in mind, we can now state and prove the fundamental theorem of the section.

Theorem 1.4.2. *Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at the origin. Let $N(f)$ be the number of Morse points which appear in a morsification of f, as discussed in the previous section. Then we have*

$$\mu(f, \underline{0}) = N(f).$$

Proof. With the Principle of Conservation of the number, the work is almost done. We take a morsification of f, that is, a deformation in the conditions of theorem 1.2.1. To that deformation, we apply the previous theorem 1.4.1. Then, for λ small enough we have

$$\mu(f, \underline{0}) = \sum_{p_i \in X_\lambda} \mu(f_\lambda, p_i).$$

Since the only critical points of f_λ are non-degenerate, we have that

$$\mu(f_\lambda, p_i) = 1$$

for every critical point p_i of f_λ. Therefore, we get what we wanted

$$\mu(f, \underline{0}) = N(f).$$

□

Chapter 2

Picard-Lefschetz Theory

In this chapter we will introduce the basic concepts of the theory of Picard-Lefschetz, which is used to investigate the topology around critical points of holomorphic functions. We begin by defining the monodromy and variation operators on a singularity. Then we study a very important example of those: the monodromy and variation operators of the Morse singularity. Afterwards, we will use the morsification which we defined in the previous chapter to give an expression of those operators for an arbitrary isolated singularity in terms of the monodromy and variation operators of Morse singularities. We will end finding basis for the homology groups in which those operators act to complete their description.

The main reference for this exposition is the second chapter of [3]

2.1 The monodromy and variation operators

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at the origin and $\mathbb{B}_\rho$ its Milnor ball. Recall the notations 1.1.1.

We know that the function

$$f^* := f|_{X^*} : X^* \to D^*$$

is locally trivial.

By the lemma 1.1.1, we also have that the restriction $f|_{\mathbb{S}_\rho \cap X} : \mathbb{S}_\rho \cap X \to D$ is a submersion. Therefore, by the Ehresmann theorem (theorem 1.1.2) we conclude that the function

$$f|_{\mathbb{S}_\rho \cap X} : \mathbb{S}_\rho \cap X \to D$$

is locally trivial, and since D is contractible, that it is actually trivial.

For $z \in D^*$, the space $X_z := f^{-1}(z) \cap \mathbb{B}_\rho$ is a complex manifold with complex dimension n and boundary

$$\partial X_z = \mathbb{S}_\rho \cap X_z.$$

Furthermore, we have also seen that it has the homotopy type of a bouquet μ of spheres of real dimension n.

Let us construct the monodromy. We fix fix $z_0 \in \partial D$ a non-critical value and the loop in D^* based at z_0 defined by

$$\gamma(t) = z_0 \cdot \exp\{2\pi i t\}, \qquad t \in [0, 1];$$

which goes round the critical value once anticlockwise. Since we know that f is locally trivial on X^* and trivial on $\mathbb{S}_\rho \cap X$ we can construct a continuous family of mappings

$$\{\Gamma_t : X_{z_0} \to X, \quad t \in I\}$$

in the following way. We take a set of open sets $\{U_i : i = 1, ..., N\}$ covering the image of loop, which we call in the same way γ, and which are trivialization domains of the fibration f^*. We consider the field of tangent vectors of γ, and using the previous trivializations, we lift those vectors locally to X. Observe that, since $f|_{\mathbb{S}_\rho \cap X}$ is trivial, we can take the lift preserving this product structure in the boundary. We use a partition of unity subordinated to the covering of the path to glue the local lifts in order to obtain a smooth field of vectors on $f^{-1}(\gamma)$. An easy calculation shows that this smooth vector field still lifts the vector field $\gamma'(t)$ over the path. Integration of the field of vectors defined on $f^{-1}(\gamma)$ gives us the desired mappings.

These mappings satisfy:

1. $\Gamma_0 : X_{z_0} \to X_{z_0}$ is the identity on the manifold X_{z_0};
2. Γ_t verifies that $f(\Gamma_t(x)) = \gamma(t)$, that is, $\Gamma_t : X_{z_0} \to X_{\gamma(t)}$
3. the family is consistent with the product structure on $\mathbb{S}_\rho \cap X_z$.

Definition 2.1.1. The transformation $h := \Gamma_1 : X_{z_0} \to X_{z_0}$ is called the **geometric monodromy** of the singularity.

Recall that any continuous mapping between topological spaces induces a corresponding mapping between the homology groups of the spaces.

Definition 2.1.2. The automorphism h_* induced by the transformation h on the only non-trivial homology group of the fibre, which is $H_n(X_{z_0})$, is called the **monodromy operator** or the **algebraic monodromy** of the singularity.

Let us make two observations.

1. If we take two another path σ which is homotopic to γ and the corresponding family of mappings $\{\tilde{\Gamma}_t : X_{z_0} \to X,\ t \in [0,1]\}$, lifting the homotopy between those two paths we get that the mappings Γ_t and $\tilde{\Gamma}_t$ are homotopic as well. Since homotopic mappings induce the same homomorphism in the homology groups, we conclude that the monodromy operator is uniquely defined by the class of the loop γ in the fundamental group $\pi_1(D^*, z_0)$.
2. If we consider a loop τ in that fundamental group which is obtained as the concatenation $\tau_1 * \tau_2$ where we first go along τ_1 and then along τ_2, it is clear that the monodromy operator will be

$$h_{\tau *} = h_{\tau_2 *} \circ h_{\tau_1 *}.$$

Therefore, we see that the correspondence assigning to each class of a loop γ based at z_0 the algebraic monodromy constructed from it h^*_γ defines an action of the fundamental group $\pi_1(D^*, z_0)$ on the homology group of the fibre $H_n(X_{z_0})$, that is, it is an antihomomorphism

$$\pi_1(D^*, z_0) \to \mathrm{Aut}(H_n(X_{z_0})).$$

Now, let us study the automorphism $h^{(r)}_{\gamma *}$ induced by h_γ in the only non-trivial relative homology group $H_n(X_{z_0}, \partial X_{z_0})$. We consider a relative cycle, that is a cycle $\delta \in C_n(X_{z_0})$ such that $\partial\delta \in C_{n-1}(\partial X_{z_0})$. Since $h_\gamma|_{\partial X_{z_0}} = \mathrm{id}$, we can conclude that $\partial h_\gamma \delta = \partial\delta$. Therefore the cycle $h_\gamma\delta - \delta$ is actually an absolute cycle on X_{z_0}, and the mapping

$$\delta \to h_\gamma\delta - \delta$$

induces a homomorphism $H_n(X_{z_0}, \partial X_{z_0}) \to H_n(X_{z_0})$.

Definition 2.1.3. The previous homomorphism

$$\mathrm{var}_\gamma : H_n(X_{z_0}, \partial X_{z_0}) \to H_n(X_{z_0})$$

is called the **variation operator** of the singularity. It is also denoted by Var_f.

Let us investigate what happens with the variation operator of the class of a loop $\tau \in \pi_1(D \setminus \{z_i\}, z_0)$ which is a concatenation $\tau_1 * \tau_2$ of two classes in that same fundamental group.

It is easy to see that

$$h_{\tau *} - \mathrm{id} = h_{\tau_2 *} \circ h_{\tau_1 *} - \mathrm{id} = h_{\tau_1 *} - \mathrm{id} + h_{\tau_2 *} - \mathrm{id} + (h_{\tau_2 *} - \mathrm{id}) \circ (h_{\tau_1 *} - \mathrm{id}).$$

Therefore, the variation operator satisfies

$$\mathrm{var}_\tau = \mathrm{var}_{\tau_1} + \mathrm{var}_{\tau_2} + \mathrm{var}_{\tau_2} \circ i_* \circ \mathrm{var}_{\tau_1} \tag{2.1}$$

where $i_* : H_n(X_{z_0}) \to H_n(X_{z_0}, \partial X_{z_0})$ is the natural homomorphism induced by the inclusion. This relation will be helpful in the sections to come.

2.2 The monodromy of the Morse singularity

In this section, we study the monodromy and variation operators of the easiest type of singularity we know: the **Morse singularity**. Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ on an analytic function with an isolated non-degenerate critical point at the origin.

We begin by studying the particular cases for the lowest dimensions: $n = 0$ and $n = 1$, since they will give us some important intuitions. Afterwards, we explain the case for higher dimensions.

2.2.1 Particular cases of dimension $n = 0$ and $n = 1$

If $n = 0$ we know that, maybe after a change of coordinates, $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is simply $f(x) = x^2$. For $z \neq 0$, the Milnor fibre here consists of two points

$$X_z = \{\sqrt{z}, -\sqrt{z}\}.$$

If we consider the path $\gamma(t) = z \cdot \exp\{2\pi i t\}$ for $t \in [0, 1]$ and we lift it via f, we get the mappings

$$\Gamma_t(\sqrt{z}) = \sqrt{z} \cdot \exp\{\pi i t\}, \qquad \Gamma_t(-\sqrt{z}) = -\sqrt{z} \cdot \exp\{\pi i t\}.$$

Therefore, we see that the monodromy h in this case interchanges the points of the Milnor fibre.

If $n = 1$, again maybe under a change of coordinates we have the germ $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$ defined as

$$f(x, y) = x^2 + y^2.$$

In lemma 1.2.2, we proved that a fibre in this case

$$X_z = \{x^2 + y^2 = z\}$$

is diffeomorphic to a disk subbundle of the tangent bundle of the sphere $\mathbb{S}^1$, of real dimension $n = 1$. We represent this space as a cylinder $\mathbb{S}^1 \times [0, 1]$ since both have the same homotopy type, and the considerations we are going to maek here are defined up to homotopy equivalence. The only non-trivial homology group of the cylinder $H_1(\mathbb{S}^1 \times [0, 1])$ is generated by the class of the real circle $\mathbb{S}^1$, which we will call Δ.

Projecting the Milnor fibre onto the complex x-line, we realise it as the Riemann surface defined by the function

$$x = \sqrt{z - y^2},$$

that is, a two-fold cover ramified at the two points $\sqrt{z}$ and $-\sqrt{z}$. This surface is obtained as two copies on the complex y-line glued along the real segment $(-\sqrt{z}, \sqrt{z})$. Each copy of the y-line is homeomorphic to half a cylindre, the segment $(-\sqrt{z}, \sqrt{z})$ corresponding to the circle $\mathbb{S}^1$ we fixed before.

Let us construct the monodromy. We fix some $\alpha > 0$ in the target complex line and consider the path

$$z(t) = \alpha \cdot \exp\{2\pi i t\}, \quad t \in [0, 1].$$

Then, following what we have just explained, for each point of the path we obtain the Riemann surface associated to

$$x = \sqrt{z(t) - y^2}.$$

Therefore, as the parameter t goes through the interval $[0, 1]$ the branch points $x = \pm\sqrt{z(t)} = \pm\sqrt{\alpha} \cdot \exp\{\pi i t\}$ move around the origin $0 \in \mathbb{C}$ anticlockwise. When we arrive to $t = 1$, the branch points have interchanged but we obtain the same Riemann surface that we had at the beginning. The situation just described is depicted in figure 2.1. In the upper part we picture the Riemann surfaces associated to the Milnor fibres, with their branch points and the segment where both lines join. Then we picture the initial and ending Milnor fibres, which are the same.

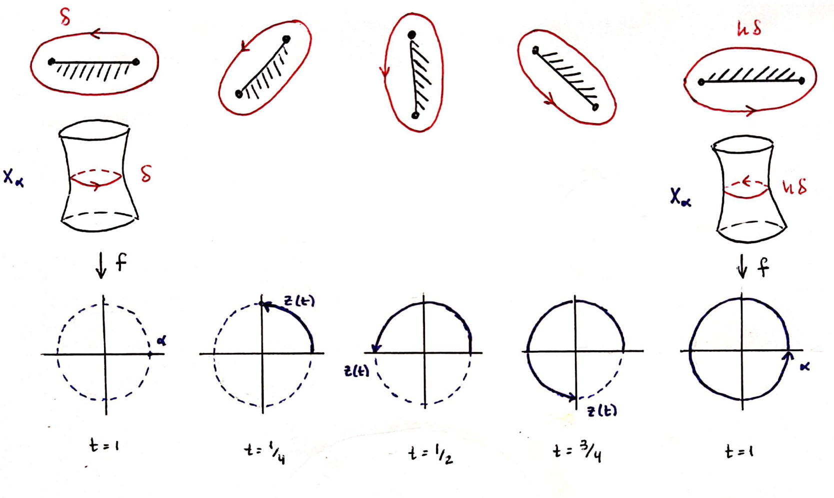

Figure 2.1: Monodromy of the germ $f(x, y) = x^2 + y^2$. The real circle δ is a generator of the class Δ generating the homology of the fibre.

Now, if we consider a circle δ going round the union of the two complex lines of the Riemann surface, its class in the homology group of the fibre will correspond to Δ and be a generator of this group. We see that, under the action of the monodromy, this class stays the same. Therefore, the monodromy acts as identity on $H_1(X_z, \mathbb{Z})$:

$$h_* = \text{Id}.$$

However, that does not mean that the monodromy is trivial. For example, we could study its action on a relative cycle τ from the relative cohomology group $H_1(X_z, \partial X_z; \mathbb{Z})$ like the one we show in figure 2.2. We find that the monodromy transforms this cycle in another which is obtained by adding the cycle we considered in the previous example $h(\tau) = \tau + \delta$. We see that the variation operator in this case acts over τ bringing it to δ. Since τ is a generator of the relative homology group $H_1(X_z, \partial X_z; \mathbb{Z})$ we also obtain that

$$\text{var} = \text{Id}.$$

Figure 2.2: Monodromy acting on a relative cycle $\tau \in H_1(X_z, \partial X_z; \mathbb{Z})$.

2.2.2 General case: Picard-Lefschetz theorem

Let us study what happens in **higher dimension**. We consider $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ the germ of an analytic function with an isolated non-degenerate critical point at the origin.

The action **variation operator** for the Morse singularity in arbitrary dimension is described by the **Picard-Lefschetz theorem**. To understand that theorem, we need to review some notions about **intersection theory**.

Let M be a manifold, and S_1 and S_2 two submanifolds of M such that $\dim S_1 + \dim S_2 = \dim M$. If we have $T_pS_1 \oplus T_pS_2 = T_pM$ for some $p \in S_1 \cap S_2$, we say that S_1 and S_2 **intersect transversally** on p. If we also suppose that M is an orientable manifold, we can define:

$$\epsilon_p := \begin{cases} +1, & \text{if } T_pS_1 \oplus T_pS_2 \text{ has the orientation of } T_pM; \\ -1, & \text{if } T_pS_1 \oplus T_pS_2 \text{ has the opposite orientation of } T_pM. \end{cases}$$

Definition 2.2.1. If S_1 and S_2 intersect transversally in every point $p \in S_1 \cap S_2$ we define the **intersection number** between S_1 and S_2 as

$$S_1 \circ S_2 := \sum_{p \in S_1 \cap S_2} \epsilon_p.$$

If we work with compact manifolds, we can show that the intersection number of two submanifolds is **well-defined** with respect to its class of **homology** in $H_*(M)$. Moreover, given two closed submanifolds of M, they can be deformed in their homology classes so that they intersect transversally in any point of their intersection. In particular, this means that self-intersection, that is, the intersection number of a closed manifold with itself, is well defined.

As a consequence of the previous observations, from now on we will talk only about the intersection number of cycles, referring to the intersection number of some representatives of those classes for which the intersection number is well defined.

Now, from **Lefschetz duality** (which is a version of Poincaré duality having in consideration the boundary of the manifold) we can conclude that if $z_0 \in \partial D$ is a non-critical value of f, the homology groups $H_k(X_{z_0}, \partial X_{z_0})$ are isomorphic to $H_k(X_{z_0})$. Therefore we have

$$H_k(X_{z_0}, \partial X_{z_0}) = 0, \quad \text{for } k \neq n; \qquad H_n(X_{z_0}, \partial X_{z_0}) = \mathbb{Z}.$$

Moreover, this duality theorem gives us a generator of the only now trivial relative homology group. We fix a cycle $\Delta \in H_n(X_{z_0})$ generating the absolute homology group. Then we can assure that $H_n(X_{z_0}, \partial X_{z_0})$ will be generated by a relative cycle ∇ dual to Δ. When we say that the cycle is dual, we mean by the duality induced by the intersection number, that is, ∇ satisfies

$$(\nabla \circ \Delta) = 1.$$

Theorem 2.2.1 (Picard-Lefschetz). *The action of the variation operator of the Morse singularity over the previous generator is the following*

$$\mathrm{Var}_f(\nabla) = (-1)^{\frac{(n+1)(n+2)}{2}} \Delta.$$

A proof for the Picarf-Lefschetz theorem can be found in section 2.4 of [3].

If we take a general relative cycle $a \in H_n(X_{z_0}, \partial X_{z_0})$, we will have $a = m \cdot \nabla$, where $m = (a \circ \Delta)$. Therefore, the variation operator acting on a will take the following form

$$\mathrm{Var}_f(a) = (-1)^{\frac{(n+1)(n+2)}{2}}(a \circ \Delta)\Delta.$$

Furthermore, recalling the relations 2.1 between the variation and the monodromy operators, we also have that for $b \in H_n(X_{z_0})$ the **monodromy operator** will act in the following manner

$$h_*(b) = b + (-1)^{\frac{(n+1)(n+2)}{2}}(b \circ \Delta)\Delta. \tag{2.2}$$

This last formula is usually called the **Picard-Lefschetz fomula**.

To understand the action of h_* over the generator $\Delta \in H_n(X_{z_0})$, we compute the **self-intersection** of this cycle. Making some easy changes of coordinates in the target disk of the Morse function f we shall suppose that $z_0 = 1$.

We know that the Milnor fibre X_1 is diffeomorphic to the disk subbundle of the tagent bundle on the standard sphere $\mathbb{S}^n$ of real dimension n (see theorem 1.2.2). Therefore, the generator Δ that we are considering can be obtained as the class of that sphere in the homology group of the fibre.

If we consider the orientation of the manifold X_1 induced by the structure of the tangent bundle of the sphere, we get that, for example at the point $(1, 0, ..., 0)$, a positively oriented coordinate system will be

$$u_2, u_3, ..., u_{n+1}, v_2, v_3, ..., v_{n+1}$$

where each complex coordinate has been descomposed as $x_j = u_j + iv_j$, $j = 1, ..., n+1$. However, if we consider its orientation as a complex manifold, a positively oriented coordinate system would be

$$u_2, v_2, u_3, v_3, ..., u_{n+1}, v_{n+1}.$$

Therefore, after a little calculation, we can see that these two orientations differ by the following sign

$$(-1)^{\frac{n(n-1)}{2}}.$$

This is relevant due to the following theorem, which we will not prove but which can be consulted in the first chapter of [3].

Theorem 2.2.2. *The self-intersection number of the zero section of the tangent bundle of a manifold coincides with the Euler characteristic χ of this manifold.*

Now, it is a fact commonly known that the Euler characteristic of the n-dimensional sphere is the following

$$\chi(\mathbb{S}^n) = 1 + (-1)^n,$$

which is equal to 0 for odd n and 2 for even n. Therefore, we can conclude that the self intersection number of Δ in X_1 oriented according to the structure of the tangent bundle is the following

$$(\Delta \circ \Delta) = 1 + (-1)^n.$$

Combining this fact with the relation between the orientations of X_1 as a subbundle and as a complex manifold, we have the result we were looking for.

Lemma 2.2.1. *The self intersection number of the vanishing cycle Δ in the complex manifold X_1 is equal to*

$$(\Delta \circ \Delta) = (-1)^{\frac{n(n-1)}{2}} [1 + (-1)^n] = \begin{cases} 0, & \text{for } n \equiv 1 \bmod 2; \\ +2, & \text{for } n \equiv 2 \bmod 4; \\ -2, & \text{for } n \equiv 0 \bmod 4. \end{cases}$$

With this lemma in mind we see that the **algebraic monodromy** acting on the generator Δ follows the same pattern that we encountered in the cases $n = 0$ and $n = 1$, that is:

- for n even, the monodromy operator acts as $h_*(\Delta) = -\Delta$,
- whereas for n odd, its action is $h_*(\Delta) = \Delta$.

2.3 Studying the monodromy from a morsification

In this section, we use a morsification $\{f_\lambda : X_\lambda \to D : |\lambda| \leq \lambda_0\}$, that is, a perturbation of f satisfying the conditions of theorem 1.2.1, to study the geometric and algebraic monodromy of a singularity as defined in section 2.1.

We fix some function f_λ from the morsification. Let $\{p_{i,\lambda} : i = 1, ..., \mu\}$ be the critical points of f_λ, where $\mu = \mu(f, \underline{0})$ and which are all non-degenerate. Let $\{z_{i_\lambda}\}$ be the critical values $z_{i,\lambda} = f(p_{i,\lambda})$, which are different. Let us fix $z_0 \in \partial D$ a non-critical value of f_λ.

We consider a set of paths $u_{i,\lambda} : [0, 1] \to D$ joining the critical value $z_{i,\lambda}$ to the non-critical value z_0, that is, satisfying $u(0) = z_{i,\lambda}$ and $u(1) = z_0$. We also suppose that these paths are non-intersecting: their only common point is z_0. Observe that in section 1.3.1 we already considered a set of paths like these. With them, we define the following loops going round the critical values.

Definition 2.3.1. A **simple loop** $\tau_{i,\lambda}$ corresponding to the path $u_{i,\lambda}$ is the class in $\pi_1(D \setminus \{z_i\}, z_0)$ of the loop going along the path $u_{i,\lambda}$ from the point z_0 to the point $z_{i,\lambda}$, going around $z_{i,\lambda}$ in the positive direction (anticlockwise) and returning to z_0 along the path $u_{i,\lambda}$ again.

Recall the notations from section 1.3.1. There, we considered the disks $\mathbb{D}_{\epsilon_i} \subset \mathbb{D}_\epsilon$ in the conditions of lemma 1.1.2 for the critical values $z_{i,\lambda}$, and called $\alpha_{i,\lambda}$ the part of the paths $u_{i,\lambda}$ joining z_0 to the boundary of $\mathbb{D}_{\epsilon_i}$. Considering the paths $\gamma_{i,\lambda}$ going along $\partial\mathbb{D}_{\epsilon_i}$ anticlockwise, we can describe the simple loop $\tau_{i,\lambda}$ as

$$\tau_{i,\lambda} = \alpha_{i,\lambda} * \gamma_{i,\lambda} * \alpha_{i,\lambda}^{-1}.$$

This situation is depicted in figure 2.3.

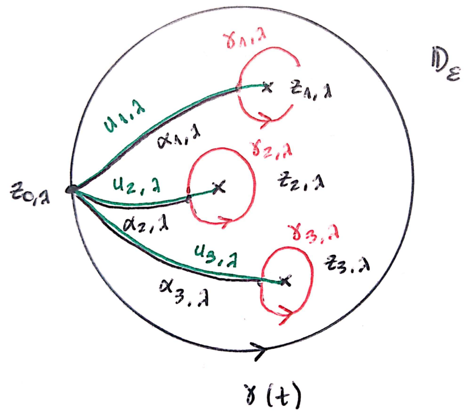

Figure 2.3: Simple loops of a morsification.

Now, we note that f_λ is locally trivial when restricted to

$$X_\lambda^* := f_\lambda^{-1}(D \setminus \{z_{i,\lambda}\}) \cap \mathbb{B}_\rho$$

and trivial when restricted to

$$\mathbb{S}_\rho \cap X_\lambda.$$

Therefore, we can define the **geometric monodromy** $h_{\gamma,\lambda}$ and **algebraic monodromy** $(h_{\gamma,\lambda})_*$ for any loop $\gamma : I \to D \setminus \{z_{i,\lambda}\}$ based at z_0 as we did in section 2.1. Using the same arguments of that section we see that if we consider a loop γ' based at z_0 homotopic to γ, then we obtain the same monodromy operators: $(h_{\gamma,\lambda})_* = (h_{\gamma',\lambda})_*$.

In particular, consider the loop upon which we constructed the monodromy of the singularity

$$\gamma(t) = z_0 \cdot \exp\{2\pi i t\}, \qquad t \in [0, 1].$$

We can use this loop to construct the monodromy of every function f_λ from the morsification since there are no critical values in D for any of those functions.

Let us see that can drop the dependence on λ on the notations for the monodromy operators for this loop. First, observe that we know that the fibres $X_{z_0,\lambda}$ are diffeomorphic for every $|\lambda| \leq \lambda_0$, so that their homology groups $H_n(X_{z_0,\lambda})$ are all isomorphic and we can identify them. The same happens with the spaces $D \setminus \{z_{i,\lambda} : i = 1, ..., \mu\}$ and their fundamental groups $\pi_1(D \setminus \{z_{i,\lambda}\}, z_0)$ for every $|\lambda| \leq \lambda_0$.

We consider the family of mappings

$$\{\Gamma_{t,\lambda} : X_{z_0,\lambda} \to X^*_\lambda, \quad t \in [0,1], \quad |\lambda| \leq \lambda_0\}$$

obtained as explained in section 2.1 for every function f_λ with $|\lambda| \leq \lambda_0$. We can actually obtain the previous family depending smoothly on the parameter λ. Then, we conclude that the mappings $\Gamma_{1,\lambda}$ are homotopic for every $|\lambda| \leq \lambda_0$. Therefore, they induce the same operators $(h_{\gamma,\lambda})_* = (h_{\gamma,\lambda'})_*$ for every $|\lambda|, |\lambda'| \leq \lambda_0$ which can be considered acting on the same homology group due to the identifications of the previous paragraph.

With these considerations in mind, for the class of the loop γ in $\pi_1(D \setminus \{z_{i,\lambda}\}, z_0)$ we call the algebraic monodromy of f_λ along that path simply $h_{\gamma*}$. What is more, we can conclude that $h_{\gamma*} = h_*$ where h_* is the algebraic monodromy of the singularity defined by $f = f_0$.

Additionally, we have that the loop γ in $\pi_1(D \setminus \{z_{i,\lambda}\}, z_0)$ is homotopic to the concatenation of the simple loops

$$\gamma \cong \tau_{1,\lambda} * \tau_{2,\lambda} * ... * \tau_{\mu,\lambda}$$

(cf. figure 2.3). Therefore, we have that

$$h_{\gamma*} = h_{\tau_{\mu,\lambda}*} \circ ... \circ h_{\tau_{2,\lambda}*} \circ h_{\tau_{1,\lambda}*}.$$

Consequently, we have proved the following characterisation of the monodromy of a singularity.

Lemma 2.3.1. *The algebraic monodromy of the singularity h_* can be computed from the monodromy operators of the simple loops in the following way*

$$h_* = h_{\tau_{1,\lambda}*} \circ h_{\tau_{2,\lambda}*} \circ ... \circ h_{\tau_{\mu,\lambda}*}.$$

Definition 2.3.2. The monodromy operators

$$h_{i*} := h_{\tau_{i,\lambda}*} : H_n(X_{z_0}) \to H_n(X_{z_0})$$

associated to the simple loops $\tau_{i,\lambda}$, for every $i = 1, ..., \mu$, are called the **Picard-Lefschetz operators of the singularity**.

We have already mentioned that the correspondence $\tau \mapsto h_{\tau *}$ is an antihomomorphism of the fundamental group $\tau \in \pi_1(D \setminus \{z_i\}, z_0)$ into the group $\mathrm{Aut}(H_n(X_{z_0,\lambda}))$ of automorphisms of the homology group $H_n(X_{z_0,\lambda})$ which is isomorphic to the homology group $H_n(X_{z_0})$.

Definition 2.3.3. The **monodromy group of the singularity** is the image of the antihomomorphism

$$\begin{array}{ccc} \pi_1(D \setminus \{z_{i,\lambda}\}, z_0) & \to & \mathrm{Aut}(H_n(X_{z_0})), \\ \tau & \mapsto & h_{\tau *}. \end{array}$$

Since the simple loops $\tau_{i,\lambda}$ form a set of generators of the group $\pi_1(D \setminus \{z_i\}, z_0)$, we can conclude that the monodromy group of the singularity will be generated by the Picard-Lefschetz operators we have just defined.

With the relation 2.1, we can also relate the variation operator of the loop γ to the variation operators of the simple loops.

Lemma 2.3.2. *The action of the variation operator of the singularity f can be defined by the following expression*

$$\begin{aligned} \mathrm{Var}_f &= \mathrm{var}_{\tau_{1,\lambda} * \tau_{2,\lambda} * \ldots * \tau_{\mu,\lambda}} \\ &= \sum_{r=1}^{\mu} \sum_{i_1 < i_2 < \ldots < i_r} \mathrm{var}_{\tau_{i_1,\lambda}} \circ i_* \circ \mathrm{var}_{\tau_{i_2,\lambda}} \circ i_* \circ \ldots \circ i_* \circ \mathrm{var}_{\tau_{i_r,\lambda}}. \end{aligned}$$

The conclusion of this section is that we can simplify the problem of studying the monodromy of a singularity to the problem of studying the action of the monodromy and variation operators associated to simple loops of a function from a morsification. Since these loops go round Morse singularities, we already know their corresponding monodromy and variation operators: they are the ones described in section 2.2. In the following section we will chose sets of generators of the homology group $H_n(X_{z_0})$ and another of the relative homology group $H_n(X_{z_0}, \partial X_{z_0})$ in order to complete this description.

2.4 Vanishing cycles and the Intersection Matrix

Recall again section 1.3.1. There, we saw how to find a family of spheres $\{\mathbb{S}_i : i = 1, \ldots, \mu\}$ in the fibres $X_{u_{i,\lambda}(t),\lambda}$ over the path $u_{i,\lambda}$, for values of the parameter $t > 0$ sufficiently close to 0. For $t = 0$ the spheres reduced to the point $p_{i,\lambda}$.

We called s_i to the point of intersection between the paths $u_{i,\lambda}$ and the disks $\mathbb{D}_{\epsilon_i}$, over which the sphere $\mathbb{S}_i$ was defined. We can carry those spheres by the monodromy along the paths $\alpha_{i,\lambda}$ and thus obtain another set of spheres

$$\{\tilde{\mathbb{S}}_i := h_{\alpha_{i,\lambda}}(\mathbb{S}_i) : i = 1, \ldots, \mu\}$$

all contained in the fibre $X_{z_0,\lambda}$ which is diffeomorphic to X_{z_0}.

Definition 2.4.1. The homology class $\Delta_i \in H_n(X_{z_0})$ corresponding to the n-dimensional sphere $\tilde{\mathbb{S}}_i$ of the previous construction is called a **vanishing** (along the path $u_{i,\lambda}$) **cycle of Picard-Lefschetz**.

It is clear for us now that this definition is unique (modulo orientation) up to the homotopy class of $u_{i,\lambda}$ in the set of paths joining $z_{i,\lambda}$ and z_0 and not passing through any other critical value of f_λ. From the construction of section 1.3.1, we see that a set of vanishing cycles $\{\Delta_i : i = 1, ..., \mu\}$ forms a **basis of the homology group** $H_n(X_{z_0})$.

Definition 2.4.2. The set of cycles $\Delta_1, ..., \Delta_\mu$ vanishing along the paths $u_{1,\lambda}, ..., u_{\mu,\lambda}$ from the homology group $H_n(X_{z_0})$ is called **distinguished** if those paths $u_{1,\lambda}, ..., u_{\mu,\lambda}$ are numbered in the same order in which they enter z_0, counted clockwise, beginning at the boundary ∂U.

For instance, the paths pictured in 2.3 give rise to a set of vanishing cycles. A basis of distinguished vanishing cycles is called **distinguished basis**. From such a basis we obtain another $\{\nabla_i : i = 1, ..., \mu\}$, this time of the relative homology $H_n(X_{z_0}, \partial X_{z_0})$ dual to $\{\Delta_i : i = 1, ..., \mu\}$, that is, such that

$$\nabla_i \circ \Delta_j = \delta_{ij}.$$

From section 2.2 we have that the matrix expression monodromy and variation operators of the singularity in these basis will be determined by the intersection of their cycles. The intersection theory in a manifold of finite homology generated by the cycles $\{\Delta_i : i = 1, ..., \mu\}$, such as the Milnor fibre X_{z_0}, can be summarised in the intersection matrix.

Definition 2.4.3. The matrix

$$S = (\Delta_i \circ \Delta_j)$$

is called the **intersection matrix**.

For the set of vanishing cycles we will call this matrix the intersection matrix of the singularity.

Definition 2.4.4. The **bilinear form associated with the singularity** is an integral bilinear form defined by the intersection number on the homology group $H_n(X_{z_0})$ of the non-singular level manifold of the function f.

The intersection matrix of the singularity is then the matrix of the bilinear form with respect to the basis of vanishing cycles $\{\Delta_i : i = 1, ..., \mu\}$. Observe that we computed its diagonal elements in lemma 2.2.1.

With all these preparations and following lemma 2.3.2, we are in the position to state the following.

Theorem 2.4.1. *The action of the variation operator of the singularity on the cycle* ∇_i *is the following*

$$\mathrm{Var}_f(\nabla_i) = (-1)^{\frac{(n+1)(n+2)}{2}} \Delta_i + \sum_{j<i} c_i^j \Delta_j$$

where c_i^j *are certain integers.*

Two important consequences follow from this theorem .

Corollary 2.4.1.1. *With respect to a distinguished basis the matrix of the variation operator* Var_f *of a singularity* f *is upper triangular matrix with diagonal entries equal to* $(-1)^{\frac{n(n-1)}{2}}$.

Therefore, Var_f is an isomorphism. Moreover, from the Picard-Lefschetz formula (equation 2.2) we have the next corollary as well.

Corollary 2.4.1.2. *The intersection matrix of a set of distinguished vanishing cycles determines the algebraic monodromy* h_*.

Chapter 3

Direct sum of singularities

In this chapter, we study how to combine two singularities in one using a construction due to Thom and Sebastiani called the *direct sum of singularities.* Then, we will investigate the topology of the singularity obtained in that construction using methods related to Picard-Lefschetz theory. We will give a description of the variation operator and the Intersection Matrix of this direct sum. We follow section 2.7 of [3].

3.1 Direct sum of singularities

Let us introduce the main object of this chapter.

Definition 3.1.1. Let $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, \underline{0}) \to (\mathbb{C}, 0)$ two singularities. We define the **direct sum** of those as the germ of the function

$$f \oplus g : (\mathbb{C}^{n+m}, \underline{0}) \to (\mathbb{C}, 0), \qquad f \oplus g\ (x, y) = f(x) + g(y).$$

Note that the direct sum of two singularities is a singularity as well, since the function we have just defined also has an isolated critical point at the origin. Concerning the Milnor number of this singularity we have the next lemma.

Lemma 3.1.1. *The Milnor number of the direct sum of the singularities* $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ *and* $g : (\mathbb{C}^m, \underline{0}) \to (\mathbb{C}, 0)$ *is*

$$\mu(f \oplus g) = \mu(f)\mu(g).$$

Proof. We can take f_λ one of the perturbations of the morsification of f and g_ν another, from the morsification of g, such that $f_\lambda + g_\nu$ is a perturbation from the morsification of $f \oplus g$. The function f_λ has $\mu(f)$ non-degenerate critical points p_i and the function g_ν has $\mu(g)$ critical points q_j. Therefore,

$f_\lambda + g_\nu$ has indeed $\mu(f)\mu(g)$ non-degenerate critical points: the ones given by the pairs (p_i, q_j). □

3.2 Homology of the join of two topological spaces

To give a description of the Variation Operator of the direct sum of two singularities, we first need to introduce the concept of join of two topological spaces and to study its homology groups.

Definition 3.2.1. Le X and Y be two topological spaces. The join $X * Y$ of those is the quotient space

$$\frac{X \times [0,1] \times Y}{\sim}$$

where $\sim$ is the equivalence relation defined by:

$$(x, 0, y_1) \sim (x, 0, y_2) \quad \text{for any} \quad y_1, y_2 \in Y,\ x \in X;$$

$$(x_1, 1, y) \sim (x_2, 1, y) \quad \text{for any} \quad x_1, x_2 \in X,\ y \in Y.$$

This construction can be seen as a space with a copy of X in its base $X \times \{0\} \times Y$ and a copy of Y at its top $X \times \{1\} \times Y$ containing all the possible non-intersecting segments joining each point of X to each point in Y.

If Y is a set consisting in only one point, the join $X * Y$ coincides with the cone over X. On the other hand, if Y is a set consisting of two points, the join is homeomorphic to the suspension of the space X.

Lemma 3.2.1. *Let us consider that the homology groups of X and Y have no torsion, or that they are being considered with coefficients in a field. Then, the reduced homology group $\tilde{H}_n(X * Y)$ of the join of those is isomorphic to*

$$\tilde{H}_n(X * Y) = \bigoplus_{0 \le k \le n-1} \tilde{H}_k(X) \otimes \tilde{H}_{n-k-1}(Y).$$

Proof. Let us see how to obtain the previous expression for the homology groups. First, we will prove that the join of the spaces $X * Y$ is homotopic to the suspension of the smash of those same spaces $X \wedge Y$. Given two points $x_0 \in X$ and $y_0 \in Y$, the smash of two topological spaces X and Y is defined as the quotient space

$$X \wedge Y = \frac{X \times Y}{\sim}$$

where $\sim$ is the equivalence relation defined by $(x, y_0) \sim (x_0, y)$ for all $x \in X$ and $y \in Y$. Therefore is we consider the space

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$$

we also have

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

If we now consider the suspension of this space, we obtain

$$S(X \wedge Y) = \frac{X \times Y \times I}{\sim}$$

where

$$\begin{array}{ll} (x, y_0, t) \sim (x_0, y, t), & \forall x \in X,\ y \in Y,\ t \in I; \\ (x_1, y_1, 0) \sim (x_2, y_2, 0) & \forall x_1, x_2 \in X,\ y_1, y_2 \in Y; \\ (x_1, y_1, 1) \sim (x_2, y_2, 1) & \forall x_1, x_2 \in X,\ y_1, y_2 \in Y. \end{array}$$

Thus, this is exactly the space $X*Y$ where we have introduced a new relation. This relation is equivalent to collapsing in $X * Y$ the space $X * \{y_0\} \cup \{x_0\} * Y$ to a point. Since $X * \{y_0\} \cup \{x_0\} * Y$ is the union of two cones, this is a contractible space, and that collapsing preserves the homotopy type. Therefore, we conclude that $X*Y$ has the same homotopy type as $S(X \wedge Y)$, and their homology groups coincide.

Now, using the Mayer-Vietoris sequence is easy to see that $\tilde{H}_n(S(X \wedge Y)) \cong \tilde{H}_{n-1}(X \wedge Y)$ for every $n \geq 1$. Combining this with the Künneth formula for relative homology we have

$$\begin{aligned} \tilde{H}_n(X * Y) &= \tilde{H}_{n-1}(X \wedge Y) = \tilde{H}_{n-1}(X \times Y, X \vee Y) = \\ &= \bigoplus_{0 \leq k \leq n-1} \tilde{H}_k(X \times Y, \{x_0\} \times Y) \otimes \tilde{H}_{n-1-k}(X \times Y, X \times \{y_0\}) = \\ &= \bigoplus_{0 \leq k \leq n-1} \tilde{H}_k(X) \otimes \tilde{H}_{n-k-1}(Y). \end{aligned}$$

□

3.3 The Variation Operator of the direct sum

To study the action of the Variation Operator of the direct sum we have to understand the homology groups of its non-singular manifold near the singular level set. This will be the first objective of this section. Then we will describe this Variation Operator.

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a singularity. Let X_{z_0} be a non-singular level manifold of the singularity near the critical point, that is

$$X_{z_0} := f^{-1}(z_0) \cap \mathbb{B}_\rho$$

for $|z_0|, \rho > 0$ sufficiently close to 0 . Let $u : [0, 1] \to \mathbb{C}$ be a path joining the non-critical value z_0 to the critical value 0.

Lemma 3.3.1. *There exists a continuous family of mappings*

$$H_t : X_{z_0} \to X_{u(t)} = f^{-1}(u(t)) \cap \mathbb{B}_\rho, \qquad \text{for } t \in [0,1]$$

such that

1. $H_0 = \mathrm{id} : X_{z_0} \to X_{z_0}$*;*
2. H_t *is the inclusion* $X_{z_0} \to X_{u(t)}$ *for* $t \in [0,1)$*;*
3. H_1 *maps* X_{z_0} *onto the point* $\underline{0} \in \mathbb{C}^n$*.*

Proof. The situation described in this proof can be pictured in the image 3.1. First, let us take a sequence $\rho = r_0 > r_1 > r_2 > ... > 0$ monotonically decreasing to 0, and the corresponding $|z_0| = \epsilon_0 > \epsilon_1 > ... > 0$ such that $f^{-1}(z)$ intersects transversely the sphere or radius r_i for every z with $|z| \leq \epsilon_i$. Those spheres are the boundary of the closed balls we are considering: $\partial \mathbb{B}_{r_i} = \mathbb{S}_{r_i}$.

From the Ehresmann theorem, we know that the function f defined between the spaces

$$E_i := f^{-1}(\mathbb{D}_{\epsilon_i}) \cap (\mathbb{B}_\rho \setminus \mathbb{B}_{r_i}) \to \mathbb{D}_{\epsilon_i}$$

is locally trivial. Since the space $\mathbb{D}_{\epsilon_i}$ is contractible, this is actually a trivial fibration. These trivializations can be chosen to coincide on

$$E_i \cap E_{i-1} = f^{-1}(\mathbb{D}_{\epsilon_i}) \cap (\mathbb{B}_\rho \setminus \mathring{\mathbb{B}}_{r_{i-1}}).$$

Now, since we can suppose the image of the path u to be contractible, we know that f is trivial when restricted to its preimage in the ball $\mathbb{B}_\rho$. Therefore, we can lift the homotopy which collapses the path u to the critical value $0 \in \mathbb{C}$ and find a family of continuous mappings $G_t : X_{z_0} \to X_{u(t)}$. This family can be chosen preserving the product structure on E_i. Since $f^{-1}(u(t)) \cap E_i$ is contractible for every $|u(t)| \leq \epsilon_i$, we can actually find a family $H_t : X_{z_0} \to X_{u(t)}$ with the properties of the lemma. □

We will use this previous lemma to give a description of the homology groups of the non-singular level manifolds of the direct sum of singularities as we announced. Let $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, \underline{0}) \to (\mathbb{C}, 0)$ be two singularities. We take representatives of those with the same target disk $\mathbb{D}_\epsilon$. For $z_0 \in \partial \mathbb{D}_\epsilon$, we call

$$X_{z_0}(f) = f^{-1}(z_0) \cap \mathbb{B}_{\rho_1}$$

and

$$X_{z_0}(g) = g^{-1}(z_0) \cap \mathbb{B}_{\rho_2}$$

the non-singular level manifolds of those singularities. Consider a path $u(t)$ non-self-intersecting in the target plane of f which joins z_0 to the critical value $0 \in \mathbb{C}$. With no loss of generality we could assume that $u(t) = (1-t)z_0$

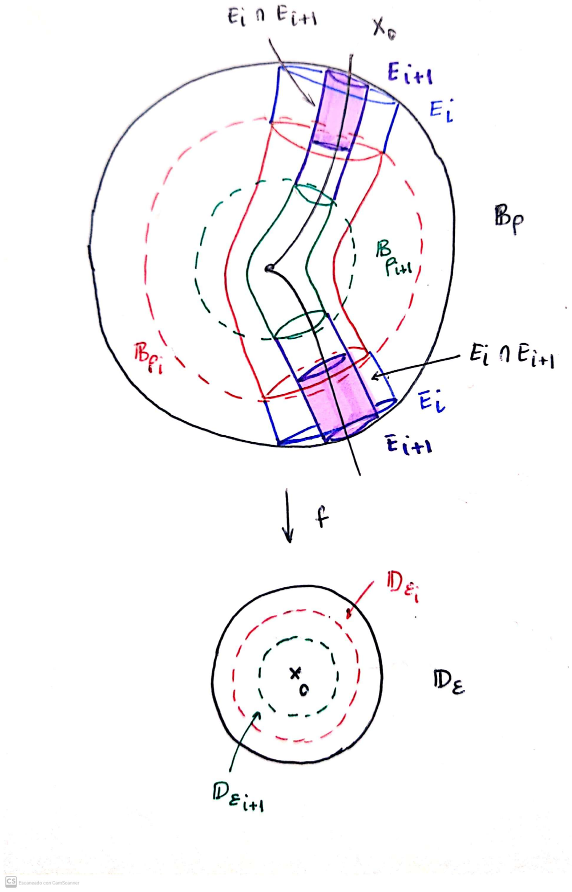

Figure 3.1: Setting for the proof of lemma 3.3.1.

with $0 \leq t \leq 1$. We also consider $v(t) = z_0 - u(1-t)$ another non-self-intersecting path, now in the target plane of g, joining z_0 and 0. Let

$$H_t(f) : X_{z_0}(f) \to X_{u(t)}(f), \qquad H_t(g) : X_{z_0}(g) \to X_{v(t)}(g)$$

be the families of functions obtained in the previous lemma.

With all these preparations we define the following inclusion of the join $X_{z_0}(f) * X_{z_0}(g)$ into the level set $(f \oplus g)^{-1}(z_0) \subset \mathbb{C}^{n+m}$:

$$j(x, t, y) = (H_t(f)(x), H_t(g)(y)).$$

Being a little careful with how we take the radii ρ_1 and ρ_2, for example taking them to be

$$\rho_1, \rho_2 \leq \frac{\rho}{2}$$

for the Milnor radius ρ of the singularity $(f \oplus g)$, we can conclude that j is an inclusion of the join $X_{z_0}(f) * X_{z_0}(g)$ into the non-singular manifold $X_{z_0}(f \oplus g) = (f \oplus g)^{-1}(z_0) \cap \mathbb{B}_\rho$.

In the previous section we described the homology groups of the join of two spaces. Since we know that the only non-trivial homology groups of $X_{z_0}(f)$ and $X_{z_0}(g)$ are $H_{n-1}(X_{z_0}(f))$ and $H_{m-1}(X_{z_0}(g))$ respectively, we have an isomorphism:

$$H_{n+m-1}(X_{z_0}(f) * X_{z_0}(g)) \cong H_{n-1}(X_{z_0}(f)) \otimes H_{m-1}(X_{z_0}(g)).$$

This isomorphism together with the inclusion

$$j : X_{z_0}(f) * X_{z_0}(g) \to X_{z_0}(f \oplus g)$$

define the homomorphism

$$j_* : H_{n-1}(X_{z_0}(f)) \otimes H_{m-1}(X_{z_0}(g)) \to H_{n+m-1}(X_{z_0}(f \oplus g)).$$

In [4] the following lemma was proved:

Lemma 3.3.2. *The homomorphism j_* is an isomorphism and the inclusion j is a homotopy equivalence.*

The previous statement allows us to identify

$$H_{n+m-1}(X_{z_0}(f \oplus g)) \cong H_{n-1}(X_{z_0}(f)) \otimes H_{m-1}(X_{z_0}(g)).$$

This identification involves another on the relative homology groups

$$H_{n+m-1}(X_{z_0}(f \oplus g), \partial X_{z_0}(f \oplus g)) \cong H_n(X_{z_0}(f), \partial X_{z_0}(f)) \otimes H_{m-1}(X_{z_0}(g), \partial X_{z_0}(g)).$$

We can now state the action of the Variation Operator of the direct sum of singularities.

Theorem 3.3.1. *Let $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, \underline{0}) \to (\mathbb{C}, 0)$ be two singularities. Then we have*

$$\mathrm{Var}_{f \oplus g} = (-1)^{nm} \mathrm{Var}_f \otimes \mathrm{Var}_g.$$

Chapter 4

The Cohomology Bundle

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function, and recall once again the notations from section 1.1.1. In the previous chapters we have studied the algebraic monodromy of the singularity. This operator allowed us to define homomorphisms between the homology groups of the Milnor fibres $H_n(X_z; \mathbb{Z})$, with $z \in D^*$, after connecting those fibres with paths in D^*. This idea of carrying one fibre to another in a fibre bundle in the motivation behind the definition of connections on bundles. In this chapter we use this approach to study the monodromy of the singularity.

In the first section, we define the cohomology bundle and describe a system of trivializations for it with the convenient feature of having locally constant transition maps. In the second section, we review the theory of connections in bundles, and use it to define a connection in the cohomology bundle from the trivializations introduced before. We introduce the notion of holonomy and conclude that the holonomy of the cohomology bundle is precisely the algebraic monodromy, as one would expect. In the third section, we introduce the horizontal sections, since the holonomy is realised by those, and characterise them with the covariant derivatives. We will end up the chapter by studying these concepts applied to the Milnor fibration of an isolated hypersurface singularity. In this situation, we will be able to define a holomorphic structure on the cohomology bundle and we will find that the bundle is analytically trivial outside the zero $0 \in \mathbb{C}$.

The references for this chapter vary from section to section, and therefore, they will be indicated at the beginning of each of those.

4.1 The cohomology bundle

As stated before, we begin by constructing the main object of the chapter: the cohomology bundle, and specifying a convenient system of trivializations of that bundle. We follow the first half of section 2 of [8], completing the details of that expostion.

Let $\pi : E \to M$ be a smooth fibre bundle, where the fibre F and the basis M are both smooth manifolds. Let us also assume that F has the homotopy type of a finite complex. Note that this is the case of the Milnor fibratio

$$\pi := f|_{X^*} : X^* \to D^*$$

with the notations of section 1.1.1.

We define a smooth complex vector bundle $\overline{\pi} : H^n(\pi) \to M$ where the total space $H^n(\pi)$ is the following set

$$H^n(\pi) := \{(z, \alpha) \in M \times H^n(X_z; \mathbb{C})\}$$

with $X_z = \pi^{-1}(z)$ and the projection is simply

$$\begin{array}{rccc} \overline{\pi} : & H^n(\pi) & \to & M \\ & (z, \alpha) & \mapsto & z \end{array}.$$

Definition 4.1.1. The complex vector bundle $\overline{\pi} : H^n(\pi) \to M$ is called the **cohomology bundle**.

The vectorial structure of the fibres is clear, since the cohomology groups are being considered with coefficients in the field of the complex numbers. Next, we point out a natural system of **local trivializations** on the cohomology bundle.

Let $\{U_i\}$ be a covering of M by open sets, each a trivialization domain of $\pi : E \to M$. Associated to those we have the transition functions

$$g_{ij} : U_i \cap U_j \to \mathrm{Aut}(F).$$

For each trivialization domain $U \subset M$, if we call $E|_U = \pi^{-1}(U)$, the diffeomorphism $\psi_U : E|_U \to U \times F$ gives us by restriction an identification $\psi_{U,z} : X_z \to F$ for every $z \in U$. Thus, we have an isomorphism

$$\psi^*_{U,z} : H^*(F; \mathbb{C}) \to H^*(X_z; \mathbb{C}).$$

With that, we define a local trivialization of $H^n(\pi)$ over U by the diffeomorphism

$$\begin{array}{rccc} \Psi_U : & H^n(\pi)|_U & \to & U \times H^n(F; \mathbb{C}) \\ & (z, \alpha) & \mapsto & \Psi_U(z, \alpha) = (z, (\psi^*_{U,z})^{-1}(\alpha)). \end{array} \tag{4.1}$$

For the previous covering, the composition

$$\begin{array}{rccc} \Psi_{U_j} \circ (\Psi_{U_i})^{-1} : & (U_i \cap U_j) \times H^n(F; \mathbb{C}) & \to & (U_i \cap U_j) \times H^n(F; \mathbb{C}) \\ & (z, \alpha) & \mapsto & (z, (\psi^*_{U_j,z})^{-1} \circ (\psi^*_{U_i,z})(\alpha)) \end{array}$$

takes the form

$$(z, \alpha) \mapsto (z, (g_{ij}(z))^* \ (\alpha))$$

since

$$(\psi^*_{U_j,z})^{-1} \circ (\psi^*_{U_i,z}) = (\psi_{U_i,z} \circ \psi^{-1}_{U_j,z})^* = (g_{ij}(z))^*$$

Note that, if we choose $V \subset U_i \cap U_j$ connected, for every $z \in V$ we have that $(g_{ij}(z))^*$ does not depend on z because all the $g_{ij}(z)$ are homotopic. Therefore, the transition functions of this trivialization of the cohomology bundle are **locally constant maps** and $\overline{\pi}$ is what is called a local system (see section 5.2). In the following sections we see that, by construction, the holonomy of $\overline{\pi}$ is the algebraic monodromy and it is realised by the locally constant sections.

4.2 Connections on bundles, locally flat connections and holonomy

In this section, we recall the definition of connection on a fibre bundle, following the approach of [5]. Then we define linear and locally flat connections, and give a characterisation of those in terms of a local matrix expression of the connection. Finally, we introduce the concept of holonomy on a fibre bundle. This last two parts are explained following [6]. As a complement, we apply all the previous concepts to the example of our interest: the cohomology bundle.

First, we note that if we have a fibre bundle $\pi : E \to M$ where π is smooth, we can define another fibre bundle $\pi_* : TE \to TM$ where

$$\begin{array}{rrcl} \pi_* : & TE & \to & TM \\ & \xi = (p, v) & \mapsto & \pi_*\xi := (\pi(p), d\pi_p v) \end{array}$$

which is well defined since $d\pi_p v \in T_{\pi(p)}M$. Having that in mind, we state the following.

Definition 4.2.1. Let $\pi : E \to M$ be a fibre bundle with π smooth. The **vertical bundle** $VE \to E$ is the subbundle of $TE \to E$ defined by

$$VE := \{\xi \in TE : \pi_*\xi = 0\} \to E.$$

Its fibres $V_pE := VE_p \subset T_pE$ are called **vertical subspaces**.

From its definition, we observe that $V_pE = T_p(E_{\pi(p)})$. Now we can introduce the definition of connection on a fibre bundle.

Definition 4.2.2. A **connection** on a fibre bundle $\pi : E \to M$ is a smooth distribution HE on E such that $HE \oplus VE = TE$, namely, for every $p \in E$ we have that $H_pE \oplus V_pE = T_pE$ where $H_pE := HE_{\pi(p)} \subset T_pE$ is called the **horizontal subspace** at p.

Example 4.2.1. Given the cohomology bundle, for every trivialization domain $U \subset M$ we encounter (cf. 4.1)

$$H^n(\pi)|_U \xrightarrow{\Psi_U} U \times H^n(F;\mathbb{C}).$$

This local product structure allows us to choose locally the horizontal and vertical distributions that define a connection: we take the pullback of the splitting of the tangent bundle of $U \times H^n(F;\mathbb{C})$ induced by the diffeomorphism Ψ_U.

Therefore, we define the horizontal subspace at $p = (z, \alpha) \in H^n(\pi)$ as

$$H_p(H^n(\pi)) := \left(d\Psi_U^{-1}\right)_{\Psi_U(p)} \left(T_z U \times \{0\}\right).$$

These local distributions glue well for different trivialization domains since the horizontal distribution is tangent to locally constant sections, such as the transition functions.

In general, let $\pi : E \to M$ be a vector bundle. We consider a system of local coordinates in $U \subset M$, so that we can see U as an open set of $\mathbb{R}^m$ and suppose that we have a local trivialization of E over U

$$E|_U \to U \times F$$

where the fibre F is also isomorphic to some real vector space $F \cong \mathbb{R}^l$. In this situation, for $p = (x, y) \in E|_U \cong U \times \mathbb{R}^l$ we can write the connection as

$$H_{(x,y)}E := \{(v, A(x,y)\ v) : v \in \mathbb{R}^m\} \tag{4.2}$$

for unique linear mappings $A(x,y) : \mathbb{R}^m \to \mathbb{R}^l$ depending smoothly on (x,y). This is what we will call the **local matrix expression** of the connection.

Definition 4.2.3. A connection is **linear** if and only if the elements of the matrix $A(x,y)$ of the previous expression depend linearly on y.

Definition 4.2.4. When the horizontal distribution is **locally flat** in the tangent bundle of the total space, we say that the connection is locally flat.

For a locally flat connection in a vector bundle $\pi : E \to M$ we can always find local matrix expressions in which the matrices $A(x,y)$ are the zero matrix.

Example 4.2.2. Let us find the matrix expression for the connection on the cohomology bundle that we defined before and conclude that it is a linear and locally flat connection.

For a trivialization domain $U \subset M$ we have

$$\begin{array}{rccl} \Psi_U^{-1}: & U \times H^n(F;\mathbb{C}) & \to & H^n(\pi) \\ & (z,\beta) & \to & (z, (\psi_{U,z})^*(\beta)) \end{array}$$

.

We know that, if U is connected, the isomorphisms $\psi_{U,z}$ are homotopic for every $z \in U$. Therefore, the induced mappings in the cohomology groups $(\psi_{U,z})^*$ do not depend on the point z (they are the same for every $z \in U$). Therefore, for $(z,\beta) \in U \times H^n(F;\mathbb{C})$ the differential

$$(d\Psi_U^{-1})_{(z,\beta)} : T_zU \times H^n(F;\mathbb{C}) \to T_{\Psi_U^{-1}(z,\beta)}H^n(\pi)$$

has as matrix

$$(d\Psi_U^{-1})_{(z,\beta)} = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & * \end{array}\right).$$

Thus, the matrix expression for the connection in this case is specially simple

$$H_p(H^n(\pi)) = \left(d\Psi_U^{-1}\right)_{(z,\beta)} (T_zU \times \{0\}) = \{(v,0) : v \in T_zU\}.$$

We see that clearly, this connection is linear. What is more, we obtain that the connection is locally flat.

Lastly, let us make some comments about the holonomy of a fibre bundle. Let $\pi : E \to M$ be a smooth bundle with smooth connection HE and let $\gamma : I \to M$ be a smooth curve in M.

Definition 4.2.5. A smooth curve $\delta : I \to E$ is called a **horizontal lift** of γ if

1. $\pi \circ \delta = \gamma$, that is, $\delta(t) \in E_{\gamma(t)}$ for every $t \in I$, and
2. for every $t \in I$ we have that $\delta'(t) \in H_{\delta(t)}E$.

From the local matrix expression of the connection 4.2 it is easy to see that for every $a \in I$ and $y \in E_{\gamma(a)}$ we can obtain a unique maximal horizontal lift δ_y of γ defined on an open interval $I_y \subset I$. We say that the connection HE **allows lifting** if for every smooth curve $\gamma : I \to M$, $a \in I$ and $y \in E_{\gamma(a)}$ there exists a horizontal lift $\delta = \delta_y : I \to E$, defined in the whole interval I, such that $\delta(a) = y$. We have the following result, whose prove can be consulted on section 4 of [6].

Lemma 4.2.1. *Every linear connection HE in a vector bundle $\pi : E \to M$ over a smooth manifold M allows lifting.*

The thing is that if a connection allows lifting, then the lifting is unique (it is the solution of an ordinary differential equation) and the mapping

$$(t,y) \mapsto \delta_y(t)$$

is a smooth mapping from $I \times E_{\gamma(a)}$ to E. Moreover we have that $\delta_y(t) \in E_{\gamma(t)}$ for every $t \in I$. The mapping

$$\begin{array}{rccc} h_\gamma^{a,t}: & E_{\gamma(a)} & \to & E_{\gamma(t)} \\ & y & \mapsto & \delta_y(t) \end{array}$$

is a smooth mapping called the **parallel transport** from the fibre $E_{\gamma(a)}$ to the fibre $E_{\gamma(t)}$ along the curve γ in M. Since $h_\gamma^{t,a}$ is a two-sided inverse of $h_\gamma^{a,t}$ we conclude that this mapping is a diffeomorphism between the fibres $E_{\gamma(a)}$ and $E_{\gamma(t)}$.

The lifting can be extended to piecewise smooth continuous curves in M. It is clear that the parallel transport along the concatenation of two curves $\gamma_1 * \gamma_2$ is equal to the composition $h_{\gamma_1} \circ h_{\gamma_2}$. If we consider a loop $\gamma : [0,1] \to M$ based at some point $x \in M$, the parallel transport along γ is then a diffeomorphism from E_x onto itself. With all these considerations in mind we see that the assignation

$$h : \gamma \mapsto h_\gamma^{0,1}$$

if a homomorphism from the group of loops based at $x \in M$ (Loop(M, x)) to the group of diffeomorphisms of the fibre E_x onto itself (Diffeo(E_x)).

Definition 4.2.6. The homomorphism h is called the **holonomy representation** of Loop(M, x) in Diffeo(E_x). The image $h(\text{Loop}(M, x)) \subset \text{Diffeo}(E_x)$ is a subgroup called the **holonomy group**.

Example 4.2.3. Let us apply this last part of the section to the connection on the cohomology bundle that we defined before. Note that this connection was in the hypothesis of lemma 4.2.1, so that we know that it allows lifting. In fact, the local flatness of this connection makes it even more clear that it allows lifting.

What is more, comparing the definition of the holonomy of that connection with the way in which we constructed the algebraic monodromy, we can conclude that the correspondence induced by the holonomy of that connection is precisely the correspondence induced by the algebraic monodromy of the the projection $\pi : E \to M$.

4.3 Horizontal sections and covariant derivatives

Let us introduce the concept of horizontal sections, define the covariant derivative along a vector field and conclude that in any locally flat connection, the horizontal sections are precisely the locally constant sections. We follow for this exposition the approach of [6].

Let $\pi : E \to M$ be a vector bundle and HE the connection on the bundle. We know that $VE = \ker d\pi$, so that for each $p \in E$ we have that

$$d\pi_p : H_pE \to T_{\pi(p)}M$$

is an isomorphism. Therefore, given a vector field $v \in \mathfrak{X}(M)$, we can define another vector field $v_{hor} \in \mathfrak{X}(E)$ such that for every $p \in E$ we have $v_{hor}(p) \in H_pE$ and such that $d\pi_p(v_{hor}(p)) = v(\pi(p))$. This is called the **horizontal lift** of the field v.

Now, if $s : M \to E$ is a smooth section of the bundle, we have $\pi \circ s = \mathrm{Id}_M$ so that $d\pi_{s(z)} \circ ds_z = \mathrm{Id}_{T_zM}$ for every $z \in M$. Moreover, given a vector field $v \in \mathfrak{X}(M)$, we know that its horizontal lift verifies: $d\pi_p(v_{hor}(p)) = v(\pi(p))$ for every $p \in E$. Therefore we can conclude that

$$d\pi_{s(z)}(ds_z(v(z))) = v(z) = d\pi_{s(z)}(v_{hor}(s(z)))$$

which means that $ds_z(v(z)) - v_{hor}(s(z)) \in \ker d\pi_{s(z)} = V_{s(z)}E$.

Definition 4.3.1. A smooth section $s : M \to E$ is **horizontal** when

$$ds_z(v(z)) - v_{hor}(s(z)) = 0 \iff ds_z(v(z)) = v_{hor}(s(z))$$

for every vector field $v \in \mathfrak{X}(M)$.

The space $V_{s(z)}E$ is the tangent space to the fibre E_z, that is, $V_{s(z)}E = T_{s(z)}E_z$. Since we are working with vector bundles, the fibres are actually vector spaces and we can identify their tangent spaces with themselves. With that, we can define the covariant derivative.

Definition 4.3.2. The **covariant derivative** of a smooth section $s : M \to E$ along the vector field $v \in \mathfrak{X}(M)$ is the following section of the bunde

$$\begin{array}{rccl} \nabla s : & M & \to & E \\ & z & \mapsto & \nabla_{v(z)} s := ds_z(v(z)) - v_{hor}(s(z)) \end{array} .$$

Observe that, from the matrix expression of the connection 4.2, it follows that $v_{hor}(s(z)) = (v, A(z, s(z))\ v)$. This leads to the following formula for the covariant derivative along a vector $v \in T_zM$ **in local coordinates** $(z_1, ..., z_m)$ in M

$$(\nabla_v s)^i(z) := \sum_{j=1}^{m} \frac{\partial s^i}{\partial z_j} v^j - \sum_{j=1}^{m} A^i_j(z, s(z)) v^j, \quad i = 1, ..., l. \tag{4.3}$$

From the previous definitions it is clear that a section $s : U \to E$, defined on an open subset $U \subset M$, satisfies $\nabla s = 0$ along any vector field in $\mathfrak{X}(U)$ if and only if s is a horizontal section on U.

Observe that the definition of horizontal section has a local nature. Note that a general connection defined on a bundle might not have any horizontal sections. Moreover, even when we find horizontal sections locally, we might not be able to extend those sections to the whole base to obtain horizontal global sections.

A particular case of connections which admit local horizontal sections are **locally flat connections**. In those, the local horizontal sections are precisely the locally constant ones. Recall that this is the case for the connection that we have defined in the cohomology bundle.

4.4 Application to isolated singularities

We will now apply the notions of the previous section to the particular case of isolated singularities of hypersurfaces. We follow the second half of section 2 of [8].

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ the germ of an analytic function with an **isolated critical point** at the origin and recall the notations from section 1.1.1. In this situation we have the Milnor fibration defined between the following spaces

$$\pi := f|_{X^*} : X^* \to D^*.$$

We know that the fibre of this bundle has the homotopy type of a bouquet of $\mu = \mu(f)$ spheres of real dimension n. Therefore, its only non-trivial cohomology groups are precisely $H^n(X_z; \mathbb{C}) = \mathbb{C}^\mu$, and thus, the only non-trivial cohomologic bundle of the fibre bundle is $H^n(\pi)$, which has rank μ.

Definition 4.4.1. The complex vector bundle $H^n(\pi)$ is called the **cohomology Milnor bundle**.

Definition 4.4.2. We will call the locally flat connection defined from the trivializations on the Milnor bundle, as suggested by Brieskorn in [17], the **local trascendental** connection.

4.4.1 Holomorphic structure

The bundle and the connection that we have just described are smooth. In this section, we give them a holomorphic structure.

To do so, we need to define the holomorphic sections of $H^n(\pi)$. We state that a section $s : D^* \to H^n(\pi)$ is holomorphic if and only if for every $z \in D^*$ there exists an open set $U \subset D^*$ and $s_1, ..., s_\mu$ a basis of locally constant sections of $H^n(\pi)$ such that s in U can be written as

$$s = \sum_{j=1}^{\mu} \phi_j s_j$$

where the functions $\phi_j : U \to \mathbb{C}$ are holomorphic. In particular, the locally constant sections are holomorphic.

Note that we can define this holomorphic structure because we are working with a locally flat connection in a complex bundle over a complex manifold. This definition is well-defined since the transition functions between frames arising from horizontal sections are locally constant.

4.4.2 Analytic triviality

We have that the cohomology bundle $H^n(\pi)$ is a holomorphic vector bundle over D^* which is a Stein manifold (see section 5.6). We will show in the end of that section that this implies that the bundle is an analytically trivial vector bundle. That means that there is a global frame of holomorphic sections $s_1, ..., s_\mu$ defined in D^*. Note that this sections are not horizontal sections in general. If we were able to find a basis of horizontal global sections of the cohomology bundle, the monodromy would be trivial, and this is not true for general hypersurface singularities.

Because of the analytic triviality, we can consider the trivial vector bundle over D as an extension of this cohomology bundle to the origin. We call this extension

$$\overline{H^n(\pi)} := \oplus_{i=1}^{\mu} \mathcal{O}_D$$

where $\mathcal{O}_D$ is the ring of holomorphic functions over the disk D. Here, we are anticipating some notations that we will introduce in the next chapter.

4.4.3 Horizontal sections

Now, we can give an expression for horizontal sections in D^* as a solution of a differential system similar to 4.3. We express the condition of being a horizontal section of the bundle $H^n(\pi)$ over D^* in terms of the base of holomorphic sections on D^*.

Let $s_1, ..., s_\mu$ be the frame of global holomorphic sections of $H^n(\pi)$ defined in D^* from the condition of analytic triviality. We know that a holomorphic section s in D^* can be expressed as

$$s = \sum_{j=1}^{\mu} \phi_j s_j$$

where the functions $\phi_j : D^* \to \mathbb{C}$ are holomorphic.

Let us denote ∇ the covariant derivative along the vector field $\frac{\partial}{\partial z}$ of D^*. We introduced covariant derivatives along vector fields in real coordinates. Here we can use that we have a basis of holomorphic sections and that the connection is holomorphic (horizontal sections are holomorphic), so we are considering directly a complex coordinate of the base. If we consider the real coordinates (x, y) such that $z = x + iy$, this is only a shortened form of denoting the covariant derivative along the vector field

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

For every $j = 1, ..., \mu$ we can express the covariant derivatives along $\frac{\partial}{\partial z}$ of the sections of the basis as

$$\nabla s_j = \sum_{k=1}^{\mu} a_{kj} s_k$$

where the a_{kj} are holomorphic functions defined in D^*.

Therefore, we have, similar to 4.3, the following expression

$$\nabla s = \sum_{j=1}^{\mu} \phi_j' s_j + \sum_{j=1}^{\mu}\sum_{k=1}^{\mu} a_{kj} s_k = \sum_{k=1}^{\mu}(\phi' + \sum_{j=1}^{k} a_{kj}\phi_j) s_k.$$

Let $\Phi = (\phi_1, ..., \phi_\mu)^t$ (column vector) and A the matrix given by the coefficients (a_{kj}). Then we have the following lemma.

Lemma 4.4.1. *A holomorphic section $s = \sum_{j=1}^{\mu} \phi_j s_j$ of $H^n(\pi)$ over D^* is a horizontal section if and only if the differential equation*

$$\Phi' + A\Phi = 0 \tag{4.4}$$

is satisfied.

We observe that the coefficients of the matrix A are holomorphic functions in D^*, so they must be meromorphic functions in D.

Chapter 5

Computing the cohomology from the complex of forms

In this chapter we justify appearance of holomorphic forms in the classical study of the monodromy. The main objective of this chapter is showing that we can actually compute the singular cohomology of the Milnor fibre from the complex of holomorphic forms defined over it. We begin introducing the cohomology of sheaves from two different points of view. Then, we review the notion of a coherent sheaf and a Stein space, in order to state Cartan's Theorems A and B. Lastly, we apply all the previous ideas to reach the objective we have just stated.

We will suppose that the reader is familiar with the definitions of sheaf (and presheaf), stalk and morphism of sheaves (and presheaves). If this is not the case, those ideas can be checked in [13] or [12].

5.1 Cohomology of sheaves

A sheaf $\mathcal{F}$ defined over a topological space X is a carrier of localized information about that space. The motivation for applying techniques of cohomological algebra to sheaves is to try to get global information about X from $\mathcal{F}$.

5.1.1 Čech Cohomology

Let us start by introducing some notations concerning Čech cohomology. We will follow the approach of Chapter II, section 4 of [13].

Let $\mathcal{F}$ be a sheaf of abelian groups over a topological space X and let $\mathcal{U} = \{U_i\}$ a locally finite covering of X by open sets.

Definition 5.1.1. We define for each natural number $p \geq 0$ the space of **cochains** of order p as

$$C^p(X, \mathcal{F}, \mathcal{U}) := \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p}).$$

We have homomorphisms

$$\delta_p : C^p(X, \mathcal{F}, \mathcal{U}) \to C^{p+1}(X, \mathcal{F}, \mathcal{U})$$

sending the element whose $(i_0, \ldots, i_p)$-coordinate is $s_{i_0 \ldots i_p} \in \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p})$ to the element whose $(i_0, \ldots, i_{p+1})$-coordinate is

$$\sum_{i=0}^{p+1} (-1)^j s_{i_0 \ldots \widehat{j} \ldots i_{p+1}}|_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}.$$

These homomorphisms are called **coboundary operators.** It is easy to see that $\delta_p \circ \delta_{p-1} = 0$, so that we have defined a complex of cochains.

We call the **p-th Čech cohomology group** of $\mathcal{F}$ **associated to the cover** $\mathcal{U}$ to the quotient

$$H^p(X, \mathcal{F}, \mathcal{U}) = \frac{\ker(\delta_p)}{\operatorname{im}(\delta_{p-1})}.$$

We consider a refinement $\mathcal{V} = \{V_j\}$ of $\mathcal{U}$, that is, a locally finite covering verifying that for every V_j there exists U_i such that $V_j \subset U_i$. Then there is a group homomorphism connecting $H^p(X, \mathcal{F}, \mathcal{U})$ and $H^p(X, \mathcal{F}, \mathcal{V})$ induced by the restrictions. Therefore, we can state the following definition.

Definition 5.1.2. We define the **p-th Čech cohomology group** of $\mathcal{F}$ as the direct limit

$$\check{H}^p(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^p(X, \mathcal{F}, \mathcal{U}).$$

Leray proved that if we consider an **acyclic covering** for the sheaf, that is, a covering which satisfies

$$H^p(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{F}) = 0, \qquad q > 0, \quad \text{any } i_0, \ldots, i_p;$$

then $H^*(X, \mathcal{F}, \mathcal{U}) = \check{H}^*(X, \mathcal{F})$.

If we consider sheaves $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ over a topological space X which verify the following exact sequence

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

then there exists a long exact sequence

$$0 \to \check{H}^0(X, \mathcal{E}) \to \check{H}^0(X, \mathcal{F}) \to \check{H}^0(X, \mathcal{G}) \\ \to \check{H}^1(X, \mathcal{E}) \to \check{H}^1(X, \mathcal{F}) \to \check{H}^1(X, \mathcal{G}) \to \ldots$$

5.1.2 Alternative definition of the cohomology

There exists another approach to the definition of the cohomology of sheaves, that coincides with Čech cohomology for paracompact spaces. In this section, we will study this alternative definition. All the proofs and details that we will omit here can be consulted in Chapter II, Section 3 of [13]. We begin by recalling resolutions of sheaves.

Definition 5.1.3. A **resolution** of a sheaf $\mathcal{F}$ is an exact sequence of sheaves of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to ... \to \mathcal{F}^m \to ...,$$

which might also be denoted by

$$0 \to \mathcal{F} \to \mathcal{F}^*.$$

Now, we introduce a particular type of sheaves which are essential.

Definition 5.1.4. A sheaf $\mathcal{F}$ over a topological space X is **soft** if for any closed subset $S \subset X$ the restriction mapping

$$\mathcal{F}(X) \to \mathcal{F}(S)$$

is surjective. Namely, any section of $\mathcal{F}$ over S can be extended to a section of $\mathcal{F}$ over X.

An interesting feature of soft sheaves is that they present no obstruction to lifting global sections.

Theorem 5.1.1. *If $\mathcal{F}$ is a soft sheaf and*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is a short exact sequence of sheaves, then the induced sequence

$$0 \to \mathcal{F}(X) \to \mathcal{E}(X) \to \mathcal{G}(X) \to 0$$

is exact.

This theorem has two important consequences.

Lemma 5.1.1. *If $\mathcal{F}$ and $\mathcal{E}$ are soft and*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is exact, then $\mathcal{G}$ is soft.

Lemma 5.1.2. *If*

$$0 \to \mathcal{F}_0 \to \mathcal{F}_2 \to \mathcal{F}_3 \to ...$$

is an exact sequence of soft sheaves over X, then the induced sequence of global sections

$$0 \to \mathcal{F}_0(X) \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to ...$$

is also exact.

Given any sheaf $\mathcal{F}$ over a topological space X, there exists a long exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^2(\mathcal{F}) \to \mathcal{C}^3(\mathcal{F}) \to ...$$

which we will call **canonical resolution of** $\mathcal{F}$. We will not enter in the details of how to define the sheaves $C^i(\mathcal{F})$. We say that this is a soft canonical resolution since the sheaf $C^0(\mathcal{F})$ is soft for any sheaf $\mathcal{F}$. Moreover, when we work wit a soft sheaf $\mathcal{F}$, from the construction which we did not detail, all the spaces $C^p(\mathcal{F})$ end up being soft.

We are now in a position to give the alternative definition of the cohomology groups of a sheaf. Let $\mathcal{F}$ be a sheaf over a topological space X and let

$$0 \to \mathcal{F} \to C^*(\mathcal{F})$$

its canonical resolution. We denote the set of global sections of a sheaf $\mathcal{F}$ in the following manner

$$\Gamma(X, \mathcal{F}) := \mathcal{F}(X).$$

With that, we see that by taking global sections this sequence induces another of the form

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, C^0(\mathcal{F})) \to \Gamma(X, C^1(\mathcal{F})) \to ... \to \Gamma(X, C^p(\mathcal{F})) \to ... \tag{5.1}$$

which is a cochain complex. This is due to the fact that taking global sections is functorial in the presheaf that we are considering. We denote

$$C^*(X, \mathcal{F}) := \Gamma(X, C^*(\mathcal{F})).$$

Definition 5.1.5. We define, for $p \geq 0$ the groups

$$H^p(X, \mathcal{F}) := H^p(C^*(X, \mathcal{F}))$$

where $H^p(C^*(X, \mathcal{F}))$ is the **p-th derived group** of the previous cochain complex, that is

$$H^p(C^*) = \frac{\ker(C^p \to C^{p+1})}{\operatorname{im}(C^{p-1} \to C^p)}, \quad \text{where } C^{-1} = 0.$$

Theorem 5.1.2. *It can be proved that these cohomology groups of $\mathcal{F}$ coincide with the ones we introduced in definition 5.1.2 if X is a paracompact space.*

This second definition allows us to access very easily to some of the fundamental properties of those groups. Let us state some of the most important ones.

- For instance, we can observe that the sequence 5.1 is exact at $\Gamma(X, C^0(\mathcal{F}))$ since this $C(\mathcal{F})$ a soft sheaf. Therefore, we can assert that

$$H^0(X, \mathcal{F}) = \mathcal{F}(X).$$

- We also have that if $\mathcal{F}$ is soft, thanks to the lemmas 5.1.1 and 5.1.2, as we said we can conclude that all the sheaves $C^p(\mathcal{F})$ are soft, and therefore the sequence of global sections is exact everywhere. Thus, in this case we have

$$H^p(X, \mathcal{F}) = 0, \qquad \text{for } p > 0.$$

- We will need the following property afterwards.

  Lemma 5.1.3. *For any sheaf morphism*

  $$h : \mathcal{A} \to \mathcal{B}$$

  there is, for each $q \geq 0$, a group homomorphism

  $$h_q : H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$$

  such that

  1. $h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X)$.
  2. *h_q is the identity map if h is the identity map, for every $q \geq 0$.*
  3. *$g_q \circ h_q = (g \circ h)_q$ for all $q \geq 0$ if $g : \mathcal{B} \to \mathcal{C}$ is another sheaf morphism.*

Lastly, let us make an observation which will be important later. We defined the cohomology groups of a sheaf from the cochain complex 5.1. We obtained that complex thanks to the functorial behaviour of taking global sections. Given a point $x \in X$, taking stalks at x is a functorial operation as well defined on the set of sheaves over X . Therefore, from the canonical resolution we might define another cochain complex

$$0 \to \mathcal{F}_x \to C^0(\mathcal{F})_x \to C^1(\mathcal{F})_x \to ... \to C^p(\mathcal{F})_x \to ... \tag{5.2}$$

which allows us to define the **cohomology of the stalks**. We denote those groups as

$$H^p(\mathcal{F}_x) := \frac{\ker(C^p(\mathcal{F})_x \to C^{p+1}(\mathcal{F})_x)}{\operatorname{im}(C^{p-1}(\mathcal{F})_x \to C^p(\mathcal{F})_x)}, \qquad p \geq 0.$$

Since taking stalks is an exact functor over the set of sheaves (it preserves exact sequences) and it is also additive (it preserves the group structure of the morphisms between modules), it can be proved that it preserves the cohomology. Therefore, we can conclude that we actually have, for every $x \in X$, an isomorphism

$$H^p(X, \mathcal{F})_x \cong H^p(\mathcal{F}_x), \qquad \forall p \geq 0.$$

5.2 Alternative definition of the cohomology bundle with sheaves

We give an alternative definition for the cohomology bundle as a locally free sheaf of modules. We follow the exposition in [20].

The first thing we shall note is that there is a bijective correspondence between vector bundles over a manifold M and locally free sheaves constructed over that same space. This bijection is defined by considering the sheaves of sections (sometimes with additional properties such as being smooth of holomorphic) over the base space. A **locally free sheaf** is a sheaf over a topological space such that locally is isomorphic to a direct sum of copies of the structure sheaf. In our case, the structure sheaf will be the ring of holomorphic functions defined on the base complex manifold. Let us see which is the locally free sheaf corresponding to the cohomology bundle.

Let X and Y be two topological spaces and $f : X \to Y$ a continuous mapping. Let also $\mathcal{F}$ be a sheaf defined over X. From it, we can define a sheaf over Y, assigning to each open subset $V \subset Y$ the abelian group associated to $f^{-1}(V)$, which is an open subset of X.

Definition 5.2.1. The previous sheaf is called the **direct image sheaf**, and it is denoted by $f_*\mathcal{F}$.

The operation in the category of sheaves of abelian groups of taking the direct image constitutes a left exact functor, but not usually right exact. We can generalise it in the following way.

Definition 5.2.2. For every $q \geq 0$, we define the **direct image sheaf of order** q as the sheaf $R^q f_*\mathcal{F}$ corresponding to the presheaf

$$U \mapsto H^q(f^{-1}(U), \mathcal{F}).$$

Here we are only going to work with direct images of constant sheaves. Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with a critical point at the origin. Recall the notations from section 1.1.1 and consider a good representative of the germ $f : X \to D$, whose restriction $\pi := f|_{X^*}$ is locally trivial. Therefore, if we consider the constant sheaf of integers over X^*, denoted $\mathbb{Z}_{X^*}$, its n-th direct image

$$R^n\pi(\mathbb{Z}_{X^*})$$

is a locally constant sheaf. Locally constant sheaves are also called local systems. According to the previous definition, its stalk over $z \in D^*$ is equal to $H^n(X_z, \mathbb{Z})$. This is called the **local system of vanishing cycles**.

If we take the complexification of the previous sheaf, we get

$$H_{D^*} := \mathbb{C} \otimes_{\mathbb{Z}} R^n\pi(\mathbb{Z}_{X^*}) = R^n\pi(\mathbb{C}_{X^*})$$

which is again a local system. To convert it to a locally free sheaf, we tensor it with the ring of holomorphic functions on D^*. The resulting sheaf is precisely the one corresponding to the **holomorphic sections of the cohomology bundle** $H^n(\pi)$ which we denote by

$$\mathcal{H}^n = \mathcal{O}_{D^*} \otimes_{\mathbb{C}} H_{D^*}.$$

As we know now, this describes cohomology classes that depend holomorphically on $z \in D^*$.

Now, we will describe the connection on the cohomology bundle, which will allow us to obtain the local system of **horizontal sections**. Let z be a complex coordinate on D and the standard vector field $\frac{\partial}{\partial z}$ dual to dz. This vector field never vanishes over the space D^*. Since the base space D^* has dimension one, to understand the connection it is enough to give the action of the covariant derivative along the previous vector field:

$$\nabla : \mathcal{H}^n \to \mathcal{H}^n, \qquad g \otimes h \mapsto \frac{dg}{dz} \otimes h.$$

With that we see that the local system of horizontal sections of the cohomological bundle is identified with the local system we started with:

$$H_{D^*} := \ker(\nabla) \subset \mathcal{H}^n.$$

This is another way of seeing how the constant sections realise the monodromy of the singularity.

5.3 Constant sheaves and simplicial cohomology

We will apply the theory of cohomology of sheaves to constant sheaves. We will see that the cohomology for these sheaves is the simplicial cohomology which we are used to work with. We follow the reference [12].

Definition 5.3.1. Considering X a topological space and G an abelian group, the **constant sheaf** (with coefficients in G) is defined as the sheaf associated to the presheaf defined by $\mathcal{F}(U) = G$ for every open connected set U. This sheaf is denoted as G_X.

Concerning those sheaves we have the following theorem.

Theorem 5.3.1. *For K a simplicial complex with underlying topological space M we have that the Čech cohomology of the constant sheaf $\mathbb{Z}_M$ on M is isomorphic to the simplicial cohomology of the complex K.*

Proof. To see this we define, associated to every vertex v_α the open set $\mathrm{St}(v_\alpha)$, called *star* of v_α, given by the interior of the union of all the simplices in K having v_α as vertex.

All these sets $\mathcal{U} = \{U_\alpha = \mathrm{St}(v_\alpha)\}$ give rise to a locally finite open covering of M. The intersection

$$\bigcap_{i=0}^{p} \mathrm{St}(v_\alpha)$$

is nonempty and connected if $v_{\alpha_0 \dots \alpha_p}$ are the vertices of a p-simplex in the decomposition of K in p-simplex. Otherwise, it is empty. Therefore, attending to the definition of Čech cohomology we have that a p-cochain σ of the sheaf $\mathbb{Z}_M$ has as $(\alpha_0, \dots, \alpha_p)$ coordinate the following

$$\sigma_{\alpha_0,\dots,\alpha_p} \in \mathbb{Z}(\cap \mathrm{St}(v_\alpha)) = \begin{cases} \mathbb{Z}, & \text{if } v_{\alpha_i} \text{ span a p} - \text{simplex}, \\ 0, & \text{otherwise.} \end{cases}$$

Let us see how to associate to this p-cochain an element from the simplicial cohomology. Given that $\sigma \in C^p(\mathcal{U}, \mathbb{Z}_M)$ we can define a p-cochain σ' setting, for $\Delta = \langle v_{\alpha_0} \dots v_{\alpha_p} \rangle$ a p-simplex with vertices $v_{\alpha_0} \dots v_{\alpha_p}$, the following

$$\sigma'(\Delta) = \sigma_{\alpha_0 \dots \alpha_p}.$$

The assignation $\sigma \mapsto \sigma'$ gives an isomorphism of abelian groups

$$C^p(M, \mathbb{Z}_M, \mathcal{U}) \cong C^p(K, \mathbb{Z})$$

which commutes with the coboundary operators

$$\delta\sigma'(\langle v_{\alpha_0} \dots v_{\alpha_p} \rangle) = \sum_{i=0}^{p+1} (-1)^{i+1} \sigma'(\langle v_{\alpha_0} \dots \widehat{v_{\alpha_i}} \dots v_{\alpha_p} \rangle = (\delta\sigma)'.$$

Therefore we have actually defined an isomorphism of complexes, hence an isomorphism

$$H^*(M, \mathbb{Z}_M, \mathcal{U}) \cong H^*(K, \mathbb{Z}).$$

Since we can subdivide the complex K to make the cover $\mathcal{U}$ of M arbitrarily fine without changing $H^*(K, \mathbb{Z})$ we finally obtain the equivalence between the cohomologies we were looking for. □

The preceding argument can be applied as well if we consider other constant sheaves, with coefficients in $\mathbb{R}$ or $\mathbb{C}$ for instance, changing the coefficients for the simplicial cohomology groups consequently.

Therefore, we have that the simplicial complex cohomology of a topological space X can be computed from the cohomology of the complex constant sheaf over that space.

5.4 Abstract de Rham Theorem

Let us state an important theorem which gives a means of computing the cohomology sheaf. For this chapter we follow the exposition in **??**.

Definition 5.4.1. We say that a resolution $0 \to \mathcal{F} \to \mathcal{A}^*$ of a given sheaf $\mathcal{F}$ over a space X is **acyclic** if $H^p(X, \mathcal{A}^q) = 0$ for all $p > 0$ and $q \geq 0$.

Observe that this was exactly the situation in soft resolutions. Indeed, if A^q is a soft sheaf for every $q \geq 0$ we have seen that $H^p(X, \mathcal{A}^q) = 0$ for every $p > 0$.

With these resolutions we will be able to compute easily the cohomology groups.

Theorem 5.4.1 (Abstract de Rham Theorem). *Let $\mathcal{F}$ be a sheaf over a space X and let*

$$0 \to \mathcal{F} \to \mathcal{A}^*$$

be a resolution of $\mathcal{F}$. Then there is a natural homomorphism

$$\gamma^p : H^p(\Gamma(X, \mathcal{A}^*)) \to H^p(X, \mathcal{F})$$

where $H^p(\Gamma(X, \mathcal{A}^*))$ is the p-th derived group of the cochain complex

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{A}^*).$$

Moreover, if the resolution

$$0 \to \mathcal{F} \to \mathcal{A}^*$$

is acyclic, then γ^p is an isomorphism.

The proof of this theorem can be checked on page 59 of [13]. That proof is constructive, and from the construction of the homomorphism γ^p, we obtain the following property.

Corollary 5.4.1.1. *Let us suppose that we have the following morphisms of sheaves*

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{A}^* \\ & & \downarrow f & & \downarrow g_* \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{B}^* \end{array}$$

which commute with the morphisms of the resolutions

$$0 \to \mathcal{F} \to \mathcal{A}^*, \qquad 0 \to \mathcal{E} \to \mathcal{B}^*.$$

Then the corresponding homomorphisms $g_p : \Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{B}^p)$ *are well defined in the cohomology groups of the complexes and the following diagram is also commutative*

$$\begin{array}{ccc} H^p(\Gamma(X, \mathcal{A}^*)) & \xrightarrow{\gamma^p} & H^p(X, \mathcal{F}) \\ \downarrow g_p & & \downarrow f_p \\ H^p(\Gamma(X, \mathcal{B}^*)) & \xrightarrow{\gamma^p} & H^p(X, \mathcal{E}) \end{array}$$

where the homomorphisms f_p *are the ones described in lemma 5.1.3.*

As a consequence, if the morphism $f : \mathcal{F} \to \mathcal{E}$ of the previous corollary is an isomorphism and the resolutions

$$0 \to \mathcal{F} \to \mathcal{A}^*, \qquad 0 \to \mathcal{E} \to \mathcal{B}^*$$

are acyclic, then all the homomorphisms γ^p are isomorphisms and we get that

$$H^p(\Gamma(X, \mathcal{A}^*)) \xrightarrow{g_p} H^p(\Gamma(X, \mathcal{B}^*))$$

is an isomorphism as well. Let us see some important applications of this theorem.

Example 5.4.1. Let X be a smooth manifold of real dimension m and let $\mathcal{E}_X^p$ be the sheaf of **real-valued differential forms** of degree p on X. Then, a resolution of the constant sheaf $\mathbb{R}_X$ is the following

$$0 \to \mathbb{R}_X \xrightarrow{i} \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}_X^m \to 0, \tag{5.3}$$

where i is the inclusion of the constant functions into the sheaf of smooth functions over X and d is the exterior differentiation operator.

To check that this is indeed a resolution, we must see that it is an exact sequence. Since $d^2 = 0$ we already know that it forms a cochain complex, and therefore

$$\mathrm{im}(\mathcal{E}^{p-1} \xrightarrow{d} \mathcal{E}^p) \subset \ker(\mathcal{E}^p \xrightarrow{d} \mathcal{E}^{p+1}).$$

To see the other inclusion, we need the classical Poincaré's Lemma.

Theorem 5.4.2 (Poincaré's Lemma). *If* $A \subset \mathbb{R}^n$ *is an open star-shaped set with respect to* $\underline{0} \in \mathbb{R}^n$*, then every closed form on* A *is exact.*

The proof of this theorem can be found in page 94 of [18]. Taking local coordinates, we can bring forms defined locally in the manifold X to an open subset of $\mathbb{R}^n$, where, maybe reducing that open set in order to have an star-shaped one, we can apply that theorem. Therefore, we conclude that for every $f \in \mathcal{E}^p_X(U)$ defined in an open subset $U \subset X$ such that $df = 0$, there exists $u \in \mathcal{E}^{p-1}_X(U)$ so that $du = f$.

With that, we conclude that the induced mappings d_x, on the stalks at any $x \in X$, are exact. The exactness at the term $\mathcal{E}^0$ is an elementary result from calculus: every $f \in \mathcal{E}^0$ such that $df = 0$ must be locally constant. Therefore, the proposed resolution is an exact sequence, as desired.

This resolution is in fact **acyclic**, since the sheaves of differentiable forms $\mathcal{E}^*_X$ over X are soft sheaves. This happens due to the existence of differentiable partitions of unity, which allow us extend sections defined in closed subsets, as the definition 5.1.4 demands. Therefore, the homomorphisms given by the abstract de Rham theorem are isomorphisms and we arrive to

$$H^p(X, \mathbb{R}_X) \cong H^p(\Gamma(X, \mathcal{E}^*_X)).$$

From the previous section we know that the former cohomology group is isomorphic to the simplicial cohomology group of the manifold.

Example 5.4.2. We have yet another resolution for the previous constant sheaf $\mathbb{R}_X$ over a smooth manifold X.

Following [13], we make the construction for a constant sheaf G_X over a topological space X where G is an abelian group. Let $S^p(U, G)$ be the **group of singular cochains** in $U \subset X$ with coefficients in G. That is

$$S^p(U, G) = \mathrm{Hom}_{\mathbb{Z}}(S_p(U, \mathbb{Z}), G)$$

where $S_p(U, \mathbb{Z})$ is the abelian group of integral singular chains of degree p in U with the usual boundary map. Let

$$\delta : S^p(U, G) \to S^{p+1}(U, G)$$

denote the corresponding coboundary operator.

We define the sheaf $S^p(G)$ as the sheaf over X generated by the presheaf which assigns to every open set $U \subset X$ the previous abelian group $S^p(U, G)$. We also have an induced morphism of sheaves $\delta : S^p(G) \to S^{p+1}(G)$.

We consider an open subset $U \subset X$ homeomorphic to the unit ball in the euclidean space (we can do this since we are working with a topological manifold). Then the sequence

$$... \to S^{p-1}(U, G) \xrightarrow{\delta} S^p(U, G) \xrightarrow{\delta} S^{p+1}(U, G) \to ...$$

is exact for $p > 0$, since $\ker \delta / \mathrm{im}\ \delta$ is the classical cohomology for the unit ball, which is zero for $p > 0$. Moreover since

$$\ker(S^0(U, G) \xrightarrow{\delta} S^1(U, G)) \cong G$$

we have exactness at $p = 0$ as well.

Therefore the sequence

$$0 \to G \to S^0(G) \xrightarrow{\delta} S^1(G) \xrightarrow{\delta} S^2(G) \to ... \to S^m(G) \to ...$$

is a resolution of the constant sheaf.

We can also consider C^∞ chains in X when X is a smooth manifold, that is, linear combinations of maps $f : \Delta^p \to U$ where f is a C^∞ mapping defined on a neighbourhood of the standard p-simplex Δ^p. The results above explained still hold in this case, and therefore we also have a resolution by differentiable cochains with coefficients in G, which we denote by

$$0 \to G \to S^*_\infty(G).$$

Let us go back to X a smooth manifold and the constant sheaf $\mathbb{R}_X$ over it. We have now two resolutions of that sheaf, namely,

$$0 \to \mathbb{R}_X \to \mathcal{E}^*_X,$$
$$0 \to \mathbb{R}_X \to S^*_\infty(\mathbb{R}).$$

We are coing to give an explicit isomorphism of sheaves as in corollary 5.4.1.1 between

$$I : \mathcal{E}^*_X \to S^*_\infty(\mathbb{R}).$$

Consider, for any open subset $U \subset X$ the homomorphisms

$$I_U : \mathcal{E}^*_X(U) \to S^*_\infty(U, \mathbb{R})$$

which to every smooth form $\omega \in \mathcal{E}^p(U)$ assign the p-cochain $I_U(\omega)$ whose action over a C^∞-chain c is the following

$$I_U(\omega)(c) := \int_c \omega.$$

The commutation with the differential and coboundary operators follows from Stoke's theorem:

$$\delta I_U(\omega)(c) = I_U(\omega)(\partial c) = \int_{\partial c} \omega = \int_c d\omega = I_u(d\omega)(c).$$

Therefore it is well defined on the homology groups of both complexes:

$$I : H^p(\Gamma(X, \mathcal{E}^*_X)) \to H^p(S^*_\infty(X, \mathbb{R}))$$

It can be proved, though we will not give all the details, that the sheaves $S^p_\infty(\mathbb{R}_X)$ are soft for every $p \geq 0$. To do so, one notices that $S^0_\infty(\mathbb{R})$ is soft since its sections are assignations, for each point of X (the singular 0-chains), of a value of $\mathbb{R}$. Therefore, if we consider a closed set $S \subset X$ we can always extend by zero. The rest of the sheaves $S^p_\infty(\mathbb{R})$ are soft because they are S^0_∞-modules and a sheaf of modules over a soft sheaf of rings is a soft sheaf (see lemma 3.16 of [13]).

Therefore, we have that the resolution

$$0 \to \mathbb{R}_X \to S^*_\infty(\mathbb{R})$$

is acyclic and thus, we have the isomorphisms

$$H^p(X, \mathbb{R}_X) \cong H^p(S^*_\infty(X, \mathbb{R})), \quad p \geq 0.$$

The first conclusion we can obtain from that fact is the equivalence between the simplicial and singular cohomologies of X with coefficients in $\mathbb{R}$.

Additionally, combining this example with the previous, we have that the following groups are isomorphic

$$H^p(S^*_\infty(X, \mathbb{R})) \cong H^p(\Gamma(X, \mathcal{E}^*_X)), \qquad p \geq 0.$$

What is more, we actually know that I induces such an isomorphism. Applying the corollary 5.4.1.1 to this situation

$$\begin{array}{ccccc} & & & & \mathcal{E}^*_X \\ & & & \nearrow & \\ 0 & \longrightarrow & \mathbb{R}_X & & \Big\downarrow I \\ & & & \searrow & \\ & & & & S^*_\infty(\mathbb{R}) \end{array}$$

where the morphism f indicated there is the identity, we conclude that the morphism I is an isomorphism and therefore

$$I : H^p(\Gamma(X, \mathcal{E}^*_X)) \to H^p(S^*_\infty(X, \mathbb{R}))$$

is the isomorphism we were looking for.

What we have proved here is commonly know as **de Rham's Theorem**.

Example 5.4.3. We come back to the case of complex manifolds. Analogously to the resolution 5.3 we can obtain the following resolution for the constant complex sheaf over a complex manifold X

$$0 \to \mathbb{C}_X \xrightarrow{i} \Omega^0_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} ... \xrightarrow{\partial} \Omega^n_X \to 0. \tag{5.4}$$

The operator ∂ is the coboundary acting on the complex-valued differential forms of type (p, q) in the following way

$$\partial : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X).$$

We also have a coboundary operator $\overline{\partial}$ which acts in the complementary way

$$\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X).$$

The exterior derivative, that is, the coboundary operator of the complex-valued differential forms of degree r

$$\mathcal{E}^r_X = \sum_{p+q=r} \mathcal{E}^{p,q}_X$$

verifies $d = \partial + \overline{\partial}$.

In order to prove that the complex is a resolution, we must check that the sequence is exact. We note that $\partial = d$ when acting on holomorphic forms $\omega \in \Omega^p_X = \mathcal{E}^{p,0}_X$, since these forms verify $\overline{\partial}\omega = 0$. With that, exactness at Ω^0_X is immediate, for the same reasons of the previous example.

For the following terms we also try to imitate the argument of the previous example. Since

$$d^2 = \partial^2 + \overline{\partial}^2 + (\partial\overline{\partial} + \overline{\partial}\partial) = 0$$

we get that $\partial^2 = 0$ and the sequence 5.4 is a complex as we stated before. Therefore, we have

$$\operatorname{im}(\Omega^{p-1} \xrightarrow{\partial} \Omega^p) \subset \ker(\Omega^p \xrightarrow{\partial} \Omega^{p+1})$$

and we only have to check the other inclusion.

To do so, we would need some kind of generalisation of Poincaré's Lemma for the operators ∂ and $\overline{\partial}$. This result was obtained by Dolbeault.

Theorem 5.4.3 (Dolbeault's Lemma). *Let U be a neighbourhood of $\underline{0} \in \mathbb{C}$ and $f \in \mathcal{E}^{p,q}_X(U)$ for $q \geq 1$ such that $\overline{\partial} f = 0$. Then, there exists a neighbourhood $V \subset U$ of $\underline{0}$ and a form $g \in \mathcal{E}^{p,q-1}_X(V)$ such that $\overline{\partial} g = f$ on V.*

The proof of this lemma can be checked on page 28 of [19]. Following the same arguments, we may conclude the same for the operator ∂. This lemma gives us exactly what we want: if $f \in \Omega^p_X(U) = \mathcal{E}^{p,0}_X(U)$ for $p \geq 1$ such that $\partial f = 0$, that is, $f \in \ker(\Omega^p \to \Omega^{p+1})$ then there exists some $g \in \Omega^{p-1}_X(V)$ such that $\partial g = f$, that is, $f \in \operatorname{im}(\Omega^{p-1} \to \Omega^p)$ as we wanted.

There is actually another argument to prove that the sequence 5.4 is a resolution. This can be consulted in proposition 7.1 of [21], page 54. The proof there only involves formal integration

Now, in order to apply the Abstract the de Rham theorem to conclude that we can calculate the homology of the space from this complex of forms, we would need the resolution to be **acyclic**. However, in this case this is not as easily proved as in the previous example. The reason for this is that there are no partitions of unity for holomorphic functions, and the sheaves of holomorphic forms are not soft. Nevertheless, we have another way of reaching the conclusion that we want, but we will need to review the theory of coherent sheaves over Stein manifolds in order to obtain it.

5.5 Coherent sheaves

Let us begin with the definitions of the spaces over which we will be constructing the sheaves from now on. Let M be a complex analytic manifold, that is, a manifold with an atlas with holomorphic transition functions. Let $\mathcal{O}_M$ be the sheaf of analytic functions on M.

Definition 5.5.1. We say that $A \subset M$ is an **analytic subset** or space of M if A is closed in M and if it is defined locally as the zero set of finitely many holomorphic functions. That is, for every $a \in A$ there is a neighbourhood $U \subset M$ of a and a set of holomorphic functions $f_1, ..., f_s$ defined in U such that

$$A \cap U = \{x \in A : f_1(x) = ... = f_s(x) = 0\}.$$

Such a space has, in general, singularities. Let us see how to define the sheaf of analytic functions over A.

Definition 5.5.2. The **sheaf of ideals** of A, denoted by $\mathcal{I}_A$ is the subsheaf of $\mathcal{O}_M$ consisting of the germs of holomorphic functions on M that vanish on A.

A stalk of that sheaf $\mathcal{I}_{A,x}$ is the ideal of germs $f \in \mathcal{O}_{M,x}$ which vanish in the germ (A, x). Note that $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ if $x \notin A$.

For every $x \in A$ we let $\mathcal{O}_{A,x}$ be the ring of germs of functions on (A, x) which can be extended as germs of holomorphic functions on (M, x). Then we have a surjective morphism $\mathcal{O}_{M,x} \to \mathcal{O}_{A,x}$ whose kernel is $\mathcal{I}_{A,x}$. Thus we get

$$\mathcal{O}_{A,x} = \frac{\mathcal{O}_{M,x}}{\mathcal{I}_{A,x}}, \qquad \forall x \in A$$

or equivalently $\mathcal{O}_A = \frac{\mathcal{O}_M}{\mathcal{I}_A}|_A$.

We can use analytic sets as a model for the construction of other spaces.

Definition 5.5.3. A **complex space** X is a locally Hausdorff space, countable at infinity, together with a sheaf $\mathcal{O}_X$ of continuous functions on X which verify that there exists an open covering $\{U_\lambda\}$ of X and for each λ a homeomorphism $F_\lambda : U_\lambda \to A_\lambda$ onto an analytic set $A_\lambda \subset \mathbb{C}^{n_\lambda}$ such that the induced morphism of rings

$$\begin{array}{cccc} F_\lambda^* : & \mathcal{O}_{A_\lambda} & \to & \mathcal{O}_X|_{U_\lambda} \\ & g & \mapsto & g \circ F \end{array}$$

is an isomorphism of rings. $\mathcal{O}_X$ is called the **structure sheaf** of X.

Given the previous isomorphism, we define the holomorphic functions over the complex space X as the functions of the structure sheaf $\mathcal{O}_X$. As we know, we can define sheaves of modules over every topological space in general. In the case of a complex space, the sheaves of modules over the structure sheaf have a particular name.

Definition 5.5.4. A sheaf of $\mathcal{O}_X$-modules over a complex space is called an **analytic sheaf**.

A very important kind of analytic sheaves are those being coherent. This will be the case of the sheaves of ideals that we have just defined $\mathcal{I}_X$ for $X \subset M$ a complex subspace of a complex manifold M or the sheaves of holomorphic forms Ω_M^*. Coherent sheaves are nice to work with since the stalk at a point x gives a lot of information about the sheaf in a small open neighbourhood of x.

The definition for coherence can be introduced for sheaves in general, but we are only interested in its application to analytic sheaves. Luckily, in this case the definition can be simplified. That simplification relies heavily in the following fact.

Theorem 5.5.1 (Oka)**.** *For any complex space X, the sheaf $\mathcal{O}_X$ is coherent in the following sense: for any open set U and any morphism of sheaves $\alpha : \mathcal{O}_X^q|_U \to \mathcal{O}_X|_U$ we have a surjective isomorphism of the type*

$$\mathcal{O}_X^p|_U \to \ker(\alpha)|_U \to 0$$

for the kernel $\ker(\alpha)$ *of the morphism.*

The proof of this very important theorem of complex analytic geometry can be checked in section 6.3 of [2]. As we said, thanks to it we can introduce s simplified definition of analytic coherent sheaves.

Definition 5.5.5. An analytic sheaf $\mathcal{F}$ on a complex space X is said to be **coherent** if for each $x \in X$ there is a neighbourhood U of x such that there is an exact sequence of sheaves over U

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{F}|_U \to 0$$

for some p and q.

Now, let us delve into the idea of how in a coherent sheaf a stalk in a point determines the sheaf in a neighbourhood of that point. Let X be a complex space and $\mathcal{O}_X$ its structure sheaf.

Theorem 5.5.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves of $\mathcal{O}_X$-modules. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of $\mathcal{O}_X$-modules. Suppose that at some point $x \in X$ the map induced in the stalks*

$$\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$$

is an isomorphism. Then, there exists an open neighbourhood U of x such that for all points $p \in U$ the maps of stalks

$$\alpha_p : \mathcal{F}_p \to \mathcal{G}_p$$

are isomorphisms, that is, the sheaves $\mathcal{F}$ and $\mathcal{G}$ are isomorphic when restricted to U.

This implies, for instance, that if $\mathcal{F}$ is a coherent sheaf, given $f_1, ..., f_k \in \mathcal{F}(U)$ which generate $\mathcal{F}_x$ for $x \in U$, we have that they are also generators of $\mathcal{F}_y$ for every $y \in V$ where V is a neighbourhood of $x \in V \subset U$.

The proof of this theorem is in section 6.2 of [2]. Let us check that the sheaves of holomorphic forms Ω^*_X over a complex manifold X are coherent.

Example 5.5.1. The sheaves of holomorphic forms over a complex manifold are coherent because they are locally free $\mathcal{O}_X$ modules and because the ring of holomorphic functions $\mathcal{O}_X$ over X is coherent.

A **locally free sheaf** (as introduced in sectio 5.2)is a sheaf over a topological space such that locally is isomorphic to a direct sum of copies of the structure sheaf. That is, for every $x \in X$ there exists a neighbourhood $U \subset X$ of x such that we have the following isomorphism of $\mathcal{O}_X$-modules

$$\Omega^*_X|_U \cong \mathcal{O}_X|_U^{\oplus r}.$$

In the previous case, we say that the sheaf is locally free of rank r.

In the case of the holomorphic forms, it is very easy to verify that condition. Taking $x \in X$, we can consider a system of local complex coordinates $(z_1, ..., z_n)$ defined on a neighbourhood of $U \subset X$ of x. In those coordinates, we have that $\Omega^p_X|_U$ has as generators the following sections

$$dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \in \Omega^p_X(U), \quad \text{for} \quad i_1 < i_2 < ... < i_p.$$

Therefore, taking $r = \begin{pmatrix} p \\ n \end{pmatrix}$ we get what we wanted

$$\Omega^p_X|_U \cong \mathcal{O}_X|_U^{\oplus r}, \quad p \geq 1.$$

Lastly, from theorem 5.5.1 we know that $\mathcal{O}_X$ is a coherent sheaf. We see from definition 5.5.5 that, in the case of analytic sheaves, coherence is a local property. That is, it suffices to see that for every $x \in X$ there exists an open neighbourhood U of x such that $\Omega_X|_U$ is coherent. It is not difficult to see that the direct sum of coherent sheaves is coherent as well, and therefore, using the isomorphism

$$\Omega^p_X|_U \cong \mathcal{O}_X|_U^{\oplus r}$$

and the fact that $\mathcal{O}_X$ is coherent, we get that $\Omega^p_X|_U$ is coherent, for every $p \geq 1$. For $p = 0$ we have directly $\Omega^0_X \cong \mathcal{O}_X$, which is coherent as well.

5.6 Stein manifolds

Now, we introduce a particular type of complex manifolds: Stein manifolds and, as a generalisation of those, Stein spaces. We are interested in these spaces since the Milnor Fibres X_z, the space X, the disk D and the punctured disk D^* (these notations are defined in 1.1.1) are Stein manifolds. We follow for this presentation the references [14] and [15].

Let us make this concept precise. There are four different possible definitions for a Stein manifold, the equivalences between them being non-trivial theorems. We will focus in a definition from Grauert in terms of the existence of a plurisubharmonic function defined over the manifold with certain property. We choose this definition since it is the easiest to handle in the cases we are interested in. Consequently, we begin by introducing this kind of functions.

Let M be a complex manifold. We call d to the operator of exterior differentiation of differential real-valued forms over M, which only uses the structure of smooth manifold. Moreover, we define another operator which uses the complex structure as well. This will be, for any $\rho : M \to \mathbb{R}$, the following operator

$$d^c\rho := d\rho \circ J$$

with J the field of multiplications by i on the complex tangent bundle of the manifold. In particular this operator verifies $J^2 = -\mathrm{Id}_{TM}$.

Definition 5.6.1. A smooth real-valued function $\rho : M \to \mathbb{R}$ is **plurisubharmonic** if

$$-dd^c\rho \geq 0.$$

It is called **strictly plurisubharmonic** if one has the stronger inequality

$$-dd^c\rho > 0.$$

Let us understand correctly what those inequalities mean. We call

$$\lambda := -d^c\rho, \qquad \omega := d\lambda.$$

From the previous definition, we have that ρ is plurisubharmonic if and only if $\omega \geq 0$. This last expression has a precise meaning.

Definition 5.6.2. A smooth-real valued differential 2-form ω on M is called **non-negative**, written $\omega \geq 0$ (respectively **positive**, written $\omega > 0$) if it is J-invariant, that is

$$\omega(J(u), J(v)) = \omega(u, v)$$

for all tangent vectors u, v of M based at the same point and if

$$\omega(v, J(v)) \geq 0, \quad \text{(respectively } \omega(v, J(v)) > 0\text{)}$$

for all non-zero tangent vectors to M.

Example 5.6.1. A canonical example of a strictly smooth plurisubharmonic function over $\mathbb{C}^n$ is the euclidean norm, that is

$$\rho : \mathbb{C}^n \to \mathbb{R}, \qquad z \mapsto \|z\|^2.$$

If we consider, for every complex coordinate, the real and imaginary parts $z_j = x_j + y_j$, we have that

$$d\rho = 2(x_1, y_1, ..., x_n, y_n).$$

The expression for the operator J in the real coordinates is a diagonal matrix of n blocks of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore we have

$$\lambda = 2(y_1, -x_1, ..., y_n, -x_n)$$

and then the matrix of ω is the opposite of the matrix of J that we described earlier, multiplied by a factor of 2. It is straightforward to check that such a form is J-invariant in the sense that we established before and that $\omega(v, J(v)) > 0$ for every vector $v \in \mathbb{C}^n$.

For every complex submanifold $X \subset \mathbb{C}^n$, the restriction of the previous function $\rho|_X$ is still plurisubharmonic. This is due to the fact that the properties of J-invariance an that $\omega(v, J(v)) > 0$ are preserved when we restrict to the complex tangent spaces $T_pM \subset \mathbb{C}^n$, for every $p \in M$. In general, with the same argument, we can assert that given a plurisubharmonic function $f : X \to \mathbb{R}$ over a complex manifold X and a complex submanifold $Y \subset X$ we have that the restriction $f|_Y$ is plurisubharmonic.

Now, we are in the position to define Stein manifolds in terms of these kind of functions.

Definition 5.6.3. A complex manifold S is **Stein** if and only if there is a strictly plurisubharmonic function $\rho : S \to \mathbb{R}$ that is an exhaustion in the sense that for every $c \in \mathbb{R}$ the sublevel set

$$\{x \in S : \rho(x) < c\}$$

is relatively compact in S, that is, its closure is a compact set.

With that definition and the plurisubharmonic function given by the restriction of the euclidean norm, we see that every closed submanifold $S \subset \mathbb{C}^n$ is Stein. Indeed, the closure of the sublevel set of the euclidean norm will be a closed set of S, and since S is closed, we conclude that it is also a closed set of $\mathbb{C}^n$. Moreover, it is clearly bounded. Therefore, it is compact, as we wanted. From this we conclude that the whole space $\mathbb{C}^n$ is a Stein manifold, for instance. The closed balls $\mathbb{B}_\rho$ of any radius $\rho > 0$, as closed complex submanifolds of $\mathbb{C}^n$ are Stein manifolds as well.

Moreover, using the same arguments we might conclude that any closed submanifold of a Stein manifold is a Stein as well. From that, we also get that the Milnor fibres X_z are Stein.

Finally, using again the euclidean norm we can check with the same arguments that the filled tube X, the disk D of radius $\epsilon > 0$ and the punctured disk D^* are also Stein manifolds.

One of the great interests of Stein manifolds is that they they allow to state Cartan's Theorems A and B, which are some fundamental theorems in complex geometry. They have a more general statement than the one we present here.

Theorem 5.6.1 (Cartan's Theorem A). *Let X be a Stein manifold, and $\mathcal{F}$ be a coherent sheaf on X. Then $H^0(X, \mathcal{F})$ generates $\mathcal{F}_x$ for all $x \in X$.*

Theorem 5.6.2 (Cartan's Theorem B). *Let X be a Stein manifold, and $\mathcal{F}$ be a coherent sheaf on X. Then $H^p(X, \mathcal{F}) = 0$ for all $p \geq 1$.*

The proofs of these theorems might be checked on page 243 of [16].

Stein manifolds have very convenient properties. As one can read in [15], the idea for these manifolds emerged of a general Oka principle, saying that on Stein spaces there are only topological obstructions to solving holomorphic problems that can be cohomologically formulated. As an application of this idea, let us show that the cohomology bundle over the punctured disk D^* must be analytically trivial, as we announced in section 4.4.2. For that, we need the following theorem, extracted from [15].

Theorem 5.6.3. *The holomorphic and topological classifications for principal bundles over Stein manifolds coincide. This holds in particular for complex vector bundles.*

Due to this theorem, we only nee to check that any vector bundle with base space D^* the punctured disk must be trivial. We observe that D^* is homotopic to $\mathbb{S}^1$, so that it suffices that we prove the same for this second base space. It is easy to see that any complex vector bundle over $\mathbb{S}^1 \subset \mathbb{C}$ must be the trivial bundle.

5.7 Computing the cohomology from the complex of holomorphic forms

We finish applying everything that we have stated to prove that if X is a Stein manifold then the cohomology groups $H^*(X, \mathbb{C})$ are obtained as the cohomology groups of the complex of holomorphic forms on X, denoted by Ω_X^*, and to give an expresion of the isomorphism between the singular cohomology of X and the homology of that complex.

We know (see section 5.3) that the simplicial cohomology with complex coefficients is the same as the coholomogy of the constant complex sheaf. Additionally, we have seen in the Abstract de Rham theorem (theorem 5.4.1) that we can compute the cohomology of a sheaf from the cohomology groups of an acyclic resolution of that sheaf. We have already studied a resolution of the constant complex sheaf $\mathbb{C}_X$ involving the complex of holomorphic forms on X, namely, the resolution 5.4. Then, we only need to check that this resolution is acyclic. In example 5.5.1 we already justified that the sheaves of holomorphic forms are coherent. Then, if X is a Stein manifold, we conclude that the resolution is acyclic due to Cartan's Theorem B (theorem 5.6.2). Therefore, we arrive to the following conclusion.

Theorem 5.7.1. *If X is a Stein manifold, the cohomology groups*

$$H^p(\Gamma(X, \Omega_X^*)) \cong H^p(X, \mathbb{C}_X)$$

are isomorphic for all $p \geq 0$.

With that we can already claim that the (simplicial) cohomology of the fibres can be computed from the cohomology groups of the complex of forms.

However, we want to establish an explicit isomorphism between the singular cohomology (with complex coefficients) resolution

$$0 \to \mathbb{C}_X \to S_\infty^*(X, \mathbb{C}) \tag{5.5}$$

and the cohomology of the complex of forms (see example 5.4.3)

$$0 \to \mathbb{C}_X \to \Omega_X^* \tag{5.6}$$

To do so, we reproduce the arguments of the example 5.4.2, that is, we apply corollary 5.4.1.1.

About the first resolution, with the reasoning explained in the example 5.4.1, we conclude that $S^0_\infty(X,\mathbb{C})$ is soft, and consequently, all the sheaves $S^p_\infty(X,\mathbb{C})$ are soft as well.

Considering that holomorphic forms are in particular smooth complex-valued forms, we can use a real parametrization of $\mathbb{C}$ to define the integral of a holomorphic ω form over a cycle $c \in X$, which will result in a complex number

$$\int_c \omega \in \mathbb{C}.$$

About these integrals, we can ensure that they satisfy Stoke's theorem. We include a proof for that fact for completeness, since this was not covered in any of the references consulted.

Theorem 5.7.2 (Stoke's theorem for holomorphic functions)**.** *Given $\omega \in \Omega^p(X)$ a holomorphic form defined in X, and c a smooth p-chain on X, then we have*

$$\int_c \partial\omega = \int_{\partial c} \omega.$$

Proof. Since $\omega \in \Omega^p(X)$ we have that the real and imaginary parts of the form, that is

$$\omega = \mathrm{Re}(\omega) + i\ \mathrm{Im}(\omega) = \alpha + i\beta$$

are differential real valued forms $\alpha, \beta \in \mathcal{E}^p(X)$.

Furthermore, since holomorphic forms verify $\overline{\partial}\omega = 0$ and we know that $d = \partial + \overline{\partial}$, we have that the action of the exterior differential over holomorphic forms is the same than the action of the operator ∂. Therefore, we can conclude

$$\partial\omega = d\omega = d\alpha + i\ d\beta.$$

Now, since we write the previous integral in terms of the real forms

$$\int_c \partial\omega = \int_c d\alpha + i\ d\beta = \int_c d\alpha + i\int_c d\beta.$$

Applying the Stoke's Theorem in the last integrals, we get what we wanted. □

With these preparations we are in the position to define the following homomorphism

$$I_p : H^p(\Gamma(X, \Omega^*_X)) \to H^p(S^p_\infty(X,\mathbb{C})), \qquad [\omega] \mapsto I_p(\omega)$$

where the cochain $I_p(\omega)$ acts on any singular p-chain c of X as

$$I_p(\omega)(c) := \int_c \omega \in \mathbb{C}.$$

That morphism is well defined because the integrals verify the Stoke's theorem 5.7.2.

Then, again thanks to the corollary 5.4.1.1, and that the resolutions

$$\begin{array}{ccccc} & & & & \Omega_X^* \\ & & & \nearrow & \downarrow I \\ 0 & \longrightarrow & \mathbb{C}_X & & \\ & & & \searrow & \\ & & & & S_\infty^*(\mathbb{C}) \end{array}$$

are acyclic we conclude that the homomorphisms I_p are actually isomorphisms.

Chapter 6

Integrals of holomorphic forms over cycles in the fibres

Let $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of analytic function with an isolated critical point at the origin and ω a holomorphic form defined in $\mathbb{C}^{n+1}$. Given a vanishing cycle $\Delta(z)$ in the fibre X_z (recall notations 1.1.1), in this chapter we study how the integral

$$\int_{\Delta(z)} \omega|_{X_Z}$$

changes when z goes around the origin $0 \in \mathbb{C}$.

We will begin this study for $n = 1$ since some intuitions are easier to present in that setting. However, all the arguments can be applied to the general case as well. We will only formulate the results there.

This exposition follows chapter 10 of [3]. The first three sections here explained are based on the section 10.1 of that reference, and the last section of this chapter sums up the main results obtained on the remaining of chapter 10 of [3].

6.1 Holomorphic dependence on parameters

Let us begin detailing constructing the integrals that we will study. Let $f : (\mathbb{C}^2, \underline{0}) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated critical point at the origin. Recall sections 1.1.1 and 2.1. We consider $\omega \in \Omega^1_{\mathbb{C}^2}$ a holomorphic 1-form and choose a closed curve $\sigma(z_0) : [0, 1] \to X_{z_0}$ in a non-singular level line X_{z_0}, for example, a parametrization of a vanishing cycle Δ.

Given $\gamma : I \to D^*$ a path joining z and z_0, we can extend the definition of the previous closed curve to the fibre X_z using the trivializations which we described in section 2.1 along the path, that is, we define

$$\sigma(z) := h_\gamma(\sigma(z_0)).$$

The object that we want to study is thus

$$I(z) := \int_{\sigma(z)} \omega|_{X_z} = \int_0^1 \sigma(z)_*(\omega|_{X_z}), \quad z \in D^*.$$

An important remark about this setting is the following: the form $\omega \in \Omega^1_{\mathbb{C}^2}$ when restricted to any fibre X_z is closed. That happens because in a complex curve there are no non-zero holomorphic 2-forms. This situation has two consequences.

1. First, the integral $I(z)$ does not change if we change $\sigma(z)$ to an homologous curve $\sigma'(z)$ in the fibre X_z. Two curves σ and σ' are homologous when they are in the same class of (singular) homology $H_1(X_z;\mathbb{Z})$, that is, they verify that there exists a real 2-chain α on X_z such that $\sigma(z) - \sigma'(z) = \partial\alpha$. Therefore, by the Stoke's theorem we have that

   $$\int_{\sigma(z)} \omega|_{X_z} - \int_{\sigma'(z)} \omega|_{X_z} = \int_{\sigma(z)-\sigma'(z)} \omega|_{X_z} = \int_{\partial\alpha} \omega|_{X_z} = \int_\alpha d(\omega|_{X_z}) = 0$$

   so that

   $$\int_{\sigma(z)} \omega|_{X_z} = \int_{\sigma'(z)} \omega|_{X_z}$$

   as we wanted.

2. The integral $I(z)$ depends clearly on the homotopy type of the path γ that we chose. If we have another path $\gamma' : I \to D^*$ joining z_0 and z homotopic to γ, we saw in section 2.1 that the operators h_γ and $h_{\gamma'}$ where homotopic. Therefore, the closed curves $h_\gamma(\sigma(z))$ and $h_{\gamma'}(\sigma(z))$ are homologous and, by the previous remark, the integral $I(z)$ is the same along any of them.

With these ideas in mind, we state the main theorem of the section.

Theorem 6.1.1. *The integral $I(z)$ is an analytic function of z, the value of f over the fibre X_z, in the sense that for every $z \in D^*$ there exists $U \subset D^*$ a neighbourhood of z such that $I|_U : D^* \to \mathbb{C}$ is an analytic function.*

In general, this function will be a multi-valuate holomorphic function as we will explain later.

To prove this theorem, we represent the integral $I(z_0)$ as another integral, this time, of a meromorphic 2-form on a real surface.

This 2-form will take the following expression

$$\frac{df \wedge \omega}{f - z_0}$$

and it will be defined in the following surface. We consider in D^* a small path γ going round z_0 anticlockwise, and define the surface in X^* formed by the union of curves

$$\Gamma := \bigcup_{z \in \gamma} \sigma(z).$$

We observe that the previous form is a holomorphic form over the path, since we have excluded the fibre X_{z_0} from its definition.

We have the following lemma, which can be thought of as a generalisation of Cauchy's Integral formula. That is, it allows us to express the value of an analytic function in a point z_0 as an integral over the boundary of a small disk ∂D_{z_0} centred at that point.

Lemma 6.1.1. *We have the expression*

$$I(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{df \wedge \omega}{f - z_0} \tag{6.1}$$

Observe that if we already knew that $I(z)$ was analytic, then, applying Cauchy's Formula we would have

$$I(z_0) = \int_\gamma \frac{I(z)}{z - z_0} dz = \int_\gamma \int_{\sigma(z)} \frac{\omega|_{X_z}}{z - z_0} dz.$$

Therefore this equality is the one we look for in the following proof.

Proof. Let us prove the equality of the integrals. From the definition of Γ, applying Fubini's theorem we get that

$$\frac{1}{2\pi i} \int_\Gamma \frac{df \wedge \omega}{f - z_0} = \frac{1}{2\pi i} \int_\gamma \left(\int_{\sigma(z)} \omega|_{X_z} \right) \frac{dz}{z - z_0},$$

which equals the following expression

$$\frac{1}{2\pi i} \int_\gamma \left(\int_{\sigma(z_0)} \omega|_{X_{z_0}} \right) \frac{dz}{z - z_0} + \frac{1}{2\pi i} \int_\gamma \left(\int_{\sigma(z)} \omega|_{X_z} - \int_{\sigma(z_0)} \omega|_{X_{z_0}} \right) \frac{dz}{z - z_0}.$$

Now, if we consider γ a circle of radius tending to zero, we get that the second term of the previous sum also tends to zero. This happens because we are integrating holomorphic forms, that is, the integrals are finite, and the dependence of z is continuous. Due to the independence of the path, we conclude that this second integral must vanish. Therefore we have

$$\frac{1}{2\pi i}\int_{\gamma}\left(\int_{\sigma(z_0)}\omega|_{X_{z_0}}\right)\frac{dz}{z-z_0}=\left(\int_{\sigma(z_0)}\omega|_{X_{z_0}}\right)\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-z_0}=I(z_0)$$

as we wanted. □

Having proved that lemma, the theorem 6.1.1 follows, since the integral

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{df\wedge\omega}{f-z_0}$$

depends analytically on z_0.

Remark. With the theory we have developed, we are in the position to **describe explicitly some natural holomorphic sections** of the cohomology bundle. Let $\omega\in\Omega^1(\mathbb{C}^2)$ be the holomorphic 1-form we had before, and let $\{\Delta_i(z):i=1,...,\mu\}$ be sets of vanishing cycles generating the homology group $H_1(X_z,\mathbb{Z})$, where $\mu:=\mu(f,\underline{0})$ is the Milnor number of the singularity and depending continuously on z.

Recall the isomorphism from section 5.7 between the cohomology of the complex of holomorphic forms and the singular cohomology of a Stein manifold. The fibres X_z are Stein manifolds, and therefore, we can define the isomorphisms

$$I:H^1(\Gamma(X_z,\Omega^*_{X_z}))\to H^1(X_z;\mathbb{C}),\qquad [\alpha]\mapsto I(\alpha)$$

such that the cochain $I(\alpha)$ acts on a 1-chain c of X_z in the following way

$$I(\alpha)(c)=\int_c\alpha.$$

The vanishing cycles generate the homology group $H_1(X_z;\mathbb{Z})$ and since

$$H_1(X_z;\mathbb{C})=\mathbb{C}\otimes_{\mathbb{Z}}H_1(X_z;\mathbb{Z})$$

we get that they are still generators for the complex homology. Additionally, it is also true that

$$H^1(X_z,\mathbb{C})=H_1(X_z,\mathbb{C})^*.$$

Therefore, the expression of the cochain $I(\omega|_{X_z})$ in coordinates in the set of generators dual to the vanishing cycles is

$$I(\omega|_{X_z})=\left(\int_{\Delta_1(z)}\omega|_{X_z},...,\int_{\Delta_\mu(z)}\omega|_{X_z}\right).$$

With that, and keeping in mind that the previous integrals are holomorphic in z, we have proved that the correspondence

$$s_\omega(z) : z \mapsto \left(\int_{\Delta_1(z)} \omega|_{X_z}, ..., \int_{\Delta_\mu(z)} \omega|_{X_z} \right)$$

defines a holomorphic section on the cohomology bundle, as we wanted.

What is more, this section is **locally constant**. If we fix $z_0 \in D^*$ and consider a connected neighbourhood $U \subset D^*$ of z_0, we can find a path $\gamma : I \to U$ joining z_0 to any other point $z \in U$. Therefore, if we consider the diffeomorphism $h_\gamma : X_{z_0} \to X_z$ obtained as described in section 2.1, we have that

$$I(z) = \int_{\Delta(z)} \omega|_{X_z} = \int_{h_\gamma^{-1}\Delta(z)} h_\gamma^*(\omega|_{X_z}) = \int_{\Delta(z_0)} \omega|_{X_{z_0}} = I(z_0)$$

as we wanted. We saw in chapter 4 that these were the sections we were interested in, since they described the holonomy of the cohomology bundle.

6.2 Branches of the integrals

In the previous chapter, we studied analyticity, which is local property of the integrals

$$I(z) := \int_{\sigma(z)} \omega|_{X_z}$$

for $\sigma(z)$ a closed curve in the fibre X_z and $\omega \in \Omega^1_{\mathbb{C}^2}$.

The objective of this chapter is to study the **global properties** of our integrals, which by analytic continuation along paths in D^* end up being multi-valued holomorphic functions.

We have seen in the previous section that, using the geometric monodromy along a path $\gamma : I \to D^*$ joining z_0 and z, we can extend the definition of the closed curve $\sigma(z_0)$ to a close curve $\sigma(z)$ in X_z and therefore the definition of $I(z)$. Moreover, we saw that the homology class of the curve $\sigma(z)$ defined that way only depends on the homotopy class of the path γ we chose to define the extension. Two homotopic paths with the same endpoints lead to homologous curves in X_z, and therefore define the same value for the integral $I(z)$. In this way, we conclude that we can continue the definition of the integral along paths in D^* giving rise to a **many-valued holomorphic function** in that domain.

What is more: from the way in which we defined the continuation of I we see that the **branching** of this integral is defined by the **monodromy of the singularity**. To each homotopy class of a closed path γ based on the point

z_0 corresponds an automorphism of the fibre X_{z_0}: the monodromy h_γ that we defined in section 2.1. If $h_{\gamma *}$ is the induced operator on the homology group of the fibre $H_1(X_{z_0};\mathbb{Z})$ then the continuation of the integral along the path γ equals by definition

$$\int_{h_{\gamma *}[\sigma(z_0)]} \omega|_{X_{z_0}}$$

where $[\sigma(z_0)]$ is the homology class of $\sigma(z_0)$ (recall that the previous integrals are defined up to the homology class of the closed curves).

6.3 Expansion of the integral in series

Since the integrals

$$I(z) := \int_{\sigma(z)} \omega|_{X_z}$$

define a many-valued holomorphic function on D^*, each branch of this function in a neighbourhood of $z \in D^*$ can be expanded in a Taylor series. In this section, we prove that in a neighbourhood of the critical value $0 \in \mathbb{C}$ the integral can be expanded as a series also. This will be a series in fractional powers of z and the coefficients of the series will be polynomials in the logarithm of z. As a consequence of the presence of those logarithms, the series converges on sectors of that small neighbourhood of the critical value.

Let us formulate the theorem with precision. Let U be a neighbourhood of $0 \in \mathbb{C}$ and in that neighbourhood let the following sector

$$S := \{z \in U : a \leq \arg(z) \leq b\}.$$

For each $z \in S$ we choose a basis

$$\sigma_1(z), ..., \sigma_\mu(z)$$

of the homology group $H_1(X_z;\mathbb{Z})$ continuously depending on z, where $\mu := \mu(f, \underline{0})$ is the Milnor number of the singularity. Let h_* be the monodromy operator on the homology groups corresponding to a path going round the critical value anticlockwise.

Theorem 6.3.1. *In the indicated sector, the vector function*

$$I(z) = \left(\int_{\sigma_1(z)} \omega|_{X_z}, ..., \int_{\sigma_\mu(z)} \omega|_{X_z} \right)$$

can be expanded in the series

$$\sum_{\alpha,k} a_{k,\alpha}\ z^\alpha\ (\ln z)^k$$

verifying the following conditions.

- *The series converges if the modulus of z is sufficiently small.*
- *The coefficients $a_{k,\alpha}$ are vectors in the space $\mathbb{C}^\mu$.*
- *The real parts of the numbers α are greater than some constant.*
- *Each number α verifies that* $\exp(2\pi i\alpha)$ *is an eigenvalue of the operator h_*.*
- *A coefficient $a_{k,\alpha}$ is equal to zero if the Jordan form of h_* does not have a block of dimension $k+1$ or more associated with the eigenvalue* $\exp(2\pi i\alpha)$.

The proof of the theorem is based on the following theorem which we will not prove here. An explanation of the arguments used to prove it can be found in page 277 of [3].

Theorem 6.3.2. *In the previous setting, there exists a natural number N for which the following inequality holds in the sector S*

$$\left| \int_{\sigma_j(z)} \omega|_{X_z} \right| \leq \text{const.} \cdot |z|^{-N}, \qquad j = 1, ..., \mu.$$

To apply the previous theorem in order to prove the theorem 6.3.1 we need to take logarithms of a non-degenerate linear transformation, that is, we also need to know the following lemma.

Lemma 6.3.1. *Let A be a non-degenerate $\mu \times \mu$ matrix. Then there exists a $\mu \times \mu$ matrix B for which*

$$\exp B = A$$

where the exponentiation of the matrix is defined, as usual, via the Taylor series

$$\exp B := \sum_{n=0}^{\infty} \frac{B^n}{n!}.$$

Furthermore, we will also need to know the next fact. We say that a linear operator is **semisimple** if the space on which it acts has a basis consisting of eigenvector of the operator. We say that it is **unipotent** if all its eigenvalues are equal to 1. For every non-degenerate linear operator M there exists a unique pair of commuting operators: a semisimple operator M_s and a unipotent one M_u, for which $M = M_u M_s$, respectively called the semisimple and unipotent parts of the operator.

Proof. First, we extend by continuity the basis $\{\sigma_i(z) : i = 1, ..., \mu\}$ that we have for points $z \in S$ in the sector to values of z with arbitrary argument.

Having those basis defined, we can extend the definition of the vector function $I(z)$ to a many-valued vector function holomorphic in a small punctured neighbourhood of 0.

Let us call the matrix of the monodromy operator h_* in the basis $\{\sigma_i(z) : i = 1, ..., \mu\}$ by A, which is a non-degenerate matrix.

The cycles $\{\sigma_i(z) : i = 1, ..., \mu\}$ generate the homology group $H_1(X_z; \mathbb{Z})$ and since

$$H_1(X_z; \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H_1(X_z; \mathbb{Z})$$

we get that they are still generators for the complex homology. Additionally, it is also true that

$$H^1(X_z, \mathbb{C}) = H_1(X_z, \mathbb{C})^*.$$

Therefore, we can obtain a basis of the complex cohomology group dual to the closed curves we have fixed.

Now, if we recall the isomorphism between the cohomology of the complex of holomorphic forms and the singular cohomology of the fibre (section 5.7) we realise that the vector function $I(z)$ respresents the coordinates of the class of the form $[\omega|_{X_z}] \in H^1(X_z; \mathbb{C})$ on that dual basis.

The homomorphism h_* induces another on the dual space, which has as matrix the transpose A^T. Therefore, we conclude that the vector function, after going one round the critical value, changes in the following way

$$I(z) \mapsto I(z) \cdot A.$$

We now consider in a punctured neighbourhood of the origin the many-valued holomorphic matrix function given by

$$J(z) := \exp\left\{ \frac{-\ln(z) \cdot \ln(A)}{2\pi i} \right\},$$

where $\ln(A)$ is one of the possible values of the logarithm of the matrix A.

If we go round the critical value once, the value of z varies to $z \cdot \exp\{2\pi i\}$ and therefore, the function $J(z)$ changes to

$$\begin{aligned} J(z \cdot \exp\{2\pi i\}) = \exp&\left\{ \frac{-\ln(z \cdot \exp\{2\pi i\}) \cdot \ln(A)}{2\pi i} \right\} \\ &= \exp\left\{ \frac{-\ln(z) \cdot \ln(A)}{2\pi i} - \ln(A) \right\} = A^{-1} J(z) \end{aligned}$$

Therefore we have that the vector function

$$z \mapsto I(z) \cdot J(z)$$

is a single-valued function in a punctured neighbourhood of zero. We prove now that this function is meromorphic at zero. Due to the theorem 6.3.2 we see that it is sufficient to prove that about the coordinates of the matrix J.

Thus, we focus on describing the elements of the matrix J. Given the decomposition of A in its semisimple and unipotent parts, it is sufficient to explain how to find the elements of J if A is diagonal of unipotent.

- If A is diagonal, then J is diagonal as well, and we can easily see that the elements of its diagonal are powers of z. The exponents of those powers are
$$\alpha = \frac{-\ln(a_{ii})}{2\pi i}$$
where a_{ii} are the elements on the diagonal of A.
- If A is unipotent then the elements of the matrix $J(z)$ are polynomials in $\ln(z)$, the degrees of the polynomials being less that the dimension of the Jordan blocks.

Therefore, for arbitrary matrices A the function $J(z)$ has the form of a finite sum
$$\sum_{\alpha} z^{\alpha} \ P_{\alpha}(\ln z).$$
In this sum, each number α verifies, as we said before, that $\exp\{-2\pi i\alpha\}$ is an eigenvalue of the matrix A. Moreover, for each α, P_{α} is a polynomial in $\ln z$ with degree less that the maximum dimension of the Jordan blocks of the matrix A associated to the eigenvalue $\exp\{-2\pi i\alpha\}$.

With that, we conclude that the coefficients of the matrix $J(z)$ grow sufficiently slowly. Therefore, the vector function $I(z) \cdot J(z)$ must be meromorphic at $0 \in \mathbb{C}$. Consequently, it can be expanded in a Laurent series with a finite number of negative exponents. Multiplying that series by $J(z)^{-1}$ we obtain the theorem 6.3.1. □

6.4 General hypersurface singularities

The generalisation of the previous theory to a germ of holomorphic function $f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin is almost straightforward. The details concerning this construction migh be revised in the sections 10.2 and 10.3 of [3]. Here we will state the main results one can find there, which are up to some point intuitive due to the theory we have developed for curves. Let $f : X \to D$ be a good representative of the germ and $f^* : X^* \to D^*$ the Milnor fibration we are constantly working with.

For any holomorphic n-form $\omega \in \Omega^n_X$, the integral of the form along the classes of a basis $\{\sigma_i(z) : i = 1, ..., \mu\}$ of the homology groups $H_n(X_z; \mathbb{Z})$

continuously varying with z defines a holomorphic many-valued function on the punctured disk D^*. The branching of this function as z goes round 0 is determined by the monodromy transformation of the homology.

The theorem analogous to theorem 6.3.1 in this context is the following.

Theorem 6.4.1. *In each sector*

$$S = \{z \in D^* : a \leq \arg z \leq b\}$$

the function

$$I(z) = \left(\int_{\sigma_1(z)} \omega|_{X_z}, ..., \int_{\sigma_\mu(z)} \omega|_{X_z} \right)$$

can be extended in the series

$$\sum_{\alpha,k} a_{k,\alpha}\ z^\alpha\ (\ln z)^k$$

which verifies the following properties.

- *All the powers α are positive and are logarithms of the eigenvalues of the classical monodromy operator divided by $2\pi i$. That is,* $\exp\{2\pi i\alpha\}$ *is an eigenvalue of the monodromy.*
- *Each power k of the logarithms in the series is less than the maximum size of the Jordan block of the classical monodromy operator associated with the corresponding eigenvalue.*

There is a slight generalisation of the previous result. To understand it, we need the theory of **Leray residues**, in which we will not enter in depth. Let us simply define those residues in the context of isolated singularities, state an important property that they verify and conclude with the mentioned generalisation. The exposition of this theory is based on section 3.1 of [20]. Some of the results are also found in the aforementioned sections of [3].

We have that X is a complex manifold of complex dimension $n+1$, and X_z for every $z \in D^*$ is a complex manifold of codimension 1 in X. Let $\alpha \in \Gamma(X, \Omega_X^{n+1})$ be a meromorphic form with its poles lying on some fibre X_z. In that setting, we say that a holomorphic differential form $\varphi \in \Gamma(X_z, \Omega^n_{X_z})$ is a **residue** of α if we can write

$$\alpha = \varphi \wedge \frac{df}{f} + \alpha'$$

where $\alpha' \in \Gamma(X, \Omega_X^{n+1})$ is holomorphic as well. We denote that form by

$$\mathrm{Res}_{X_z}(\alpha) = \varphi|_{X_z}.$$

Let us state the generalisation of lemma 6.1.1 which is verified by those residues.

Lemma 6.4.1. *The integral of the residue of a form* $\alpha \in \Gamma(X, \Omega_X^{n+1})$ *over some cycle* $\sigma \in H_n(X_z, \mathbb{Z})$ *equals the following integral*

$$\int_\sigma \mathrm{Res}_{X_z}(\alpha) = \frac{1}{2\pi i} \int_\Delta \alpha$$

where $\Delta \subset X \setminus X_z$ *is obtained replacing each point of* δ *by a small circle encircling the fibre* X_z *in the positive direction. Observe that the second integral is well defined because we excluded the fibre* X_z *were* α *had its poles.*

With these ideas in mind, we have the following. Let $\omega \in \Gamma(X, \Omega_X^{n+1})$ be a holomorphic form as usual. We fix $z \in D^*$ and consider the holomorphic form on the fibre X_z given by the following residue

$$\eta_z := \mathrm{Res}_{X_z}\left(\frac{\omega}{f-z}\right).$$

We also consider the integrals

$$I(z) := \int_{\sigma(z)} \eta_z = \frac{1}{2\pi i} \int_{\Delta(z)} \frac{\omega}{f - z'}$$

where $\sigma(z) \in H_n(X_z, \mathbb{Z})$. It is not difficult to convince oneself that we can transport the cycles $\sigma(z)$ to nearby fibres and therefore obtain a well-defined holomorphic function in a neighbourhood of each $z \in D^*$.

The generalisation of theorem 6.3.1 which we mentioned is then stated for these integrals.

Theorem 6.4.2. *All the integrals of the form*

$$I(z) := \int_{\sigma(z)} \mathrm{Res}_{X_z}\left(\frac{\omega}{f-z}\right)$$

in the previous setting can be expanded in each sector

$$S = \{z \in D^* : a \leq \arg z \leq b\}$$

in a series of the form

$$\sum_{\alpha,k} a_{k,\alpha}\; z^\alpha \;(\ln z)^k$$

which verifies the following properties.

- *It converges in the modulus of* z *is sufficiently small.*
- *All the powers* α *verify* $-1 < \alpha$*.*
- *Each number* $\exp\{2\pi i\alpha\}$ *is an eigenvalue of the monodromy.*
- *The coefficients* $a_{k,\alpha}$ *are equal to zero at any time that the classical monodromy operator does not have Jordan blocks of dimension* $k+1$ *or greater associated with the eigenvalue* $\exp\{2\pi i\alpha\}$*.*

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