Understanding the proof of Lawson's conjecture

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1 Introduction

Minimal surfaces in spaces of constant curvature have historically been of great interest to differential geometers. The subject has its origins in the study of area-minimisers in $\mathbb{R}^3$, but an arguable defect of the theory in flat geometries is that it admits no closed examples. If we pass to the spherical case however, these exist (in low dimensions at least) abundantly: it was proved recently by Marques-Neves in [16] that any closed Ricci-positive manifold of dimension $3 \leq n \leq 7$ contains infinitely many embedded minimal hypersurfaces.

Of course, this impressive existence result does not imply that explicit examples are easy to write down, and little is known generally about how to construct them. One situation where we can actually be concrete is that of the 3-dimensional sphere $\mathbb{S}^3$, and it is on this case that our project will exclusively focus.

The search for minimal surfaces in $\mathbb{S}^3$ begins with the equators. These are the 2-spheres of maximal radius, and they are not only minimal but totally geodesic, which becomes obvious upon noting that the geodesics in any sphere are the great circles. A more thorough search yields the Clifford torus, which can be defined, after identifying $\mathbb{S}^3$ with the unit sphere contained in $\mathbb{R}^4$, by

$$\left\{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\right\}. \tag{1}$$

This is clearly embedded. Moreover one can check that its principal curvatures are ± 1, so its mean curvature vanishes identically (in fact so does its Gauss curvature, cf. Appendix A) and thus it is minimal. The Clifford torus is probably the most natural and symmetric torus one could consider in $\mathbb{S}^3$ (it is, for example, fibered over the real equator in $\mathbb{S}^2$ by the Hopf map), and though Lawson proved in [14] that it has infinitely many minimally immersed siblings, he was unable to find another embedded example of genus 1. In fact, he was led to formulate in [13] the following conjecture.

Lawson's conjecture (1970): the Clifford torus is the unique minimal embedded torus in the 3-sphere up to congruence.[1]

Lawson produced a fair amount of evidence for the truth of his conjecture. Indeed, also in [13] he was able to show that any embedded minimal torus may be deformed onto the Clifford torus by an ambient diffeomorphism. Furthermore in the earlier [15] he showed that a sufficient condition for a torus being Clifford was having a parallel second fundamental form. This, in combination with work of Simons in [17] on the rigidity properties of the second fundamental form for minimal surfaces in spheres, pointed to some concrete strategies one could use to attack the conjecture.

In 2012, after many partial results and attempted solutions, Lawson's question was answered in the affirmative by Brendle. His proof is given in [5], and the primary aim of this text is to

[1] Two surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{S}^3$ are said to be congruent if there exists an isometry $f : \mathbb{S}^3 \to \mathbb{S}^3$ such that $f(\Sigma_1) = \Sigma_2$.

give an overview of his argument, filling in the details that are not explicitly contained in his paper. We also hope to shed light on the technical results to which his methods appeal, and to explain the origins and geometric intuitions of the proof. The text will have the following structure.

- **Section 2** will run through an abridged and slightly rearranged form of Brendle's proof, attempting to keep the "story" in view at all times.
- In **section 3** we describe, and in some cases prove, the key results which make Brendle's work tick. We particularly focus on his innovative application of a certain degenerate maximum principle.
- **Section 4** is concerned with the idea and the origin of Brendle's argument, and attempts to frame it in light of a similar technique used by Andrews in the context of the mean curvature flow. We hope in particular to motivate the functions and quantities of interest in Brendle's proof.
- The brief **section 5** will speak on the state of natural generalisations of Lawson's conjecture.
- Finally, the **appendices** include a number of more elementary/mindnumbing calculations which will be invoked by the main text at various times.

2 A review of Brendle's argument

In this section we outline in detail the argument used by Brendle in [5] to prove Lawson's conjecture. We take this opportunity to fix the following notation:

- $F : \Sigma \to \mathbb{S}^3 \subset \mathbb{R}^4$ is an embedded minimal torus (we will frequently abuse this and identify $F(\Sigma)$ with Σ);
- $\nu : \Sigma \to \mathbb{R}^4$ is a choice of unit normal to Σ, tangent to the sphere $\mathbb{S}^3$;
- $h \in \Gamma(\mathrm{Sym}^2 T^*\Sigma)$ is the second fundamental form of F with respect to ν;
- $|A| : \Sigma \to \mathbb{R}$ is the norm of the shape operator of F with respect to ν;
- $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^4$;
- ∇^0, ∇^S and ∇^Σ are the Levi-Civita connections on $\mathbb{R}^4$, $\mathbb{S}^3$ and Σ respectively.

We also recall the fact that $h(X,Y) = -\langle \nabla^S_X \nu, Y\rangle = -\langle \nabla^0_X \nu, Y\rangle$.

Brendle's proof of Lawson's Conjecture is organised around an understanding of the quantity

$$\kappa := \sup_{x,y\in\Sigma,\ x\neq y} \sqrt{2}\frac{|\langle \nu(x), F(y)\rangle|}{|A(x)|(1-\langle F(x), F(y)\rangle)}. \tag{2}$$

The significance of this is fully explored in §4.2, but for the moment it can be understood as capturing how much the maximum principal curvature of the torus deviates from the curvature of the largest possible ball inscribed within it. The fact that this supremum is well-defined follows from both of the assumptions in the statement of Lawson's conjecture.

1. In [14] Lawson proved that a minimal torus in $\mathbb{S}^3$, such as Σ, has no umbilic points (i.e. points in which the principal curvatures are the same). We include that proof, and some interesting remarks associated to it, in section 3.1. This fact implies that $|A| \neq 0$ in Σ, since otherwise we would need both the principal curvatures to vanish.
2. On the other hand, the embeddedness gives us that the denominator only vanishes when $x = y$. We will see next that, in that case, the numerator also vanishes in a way that makes the supremum be well defined.

As we have just stated, let us prove the following key property of the quantity κ.

Proposition 2.1. *The supremum κ is always greater or equal than 1. Moreover, if $\kappa > 1$, then the supremum is attained outside of the diagonal.*

Proof. Let us fix some $x \in \Sigma$. We can assume that our parametrization F around x is an exponential chart

$$\exp_x : U \to \Sigma$$

defined on an open subset $U \subset T_x\Sigma$, such that $F(0) = x$. We also consider our unit normal vector defined on that open set, that is, $\nu : U \to \mathbb{R}^4$.

Using that $|F| \equiv 1$ we have

$$\langle F(v) - F(0), F(v) - F(0)\rangle = 2(1 - \langle F(0), F(v)\rangle)$$

so that, also taking into account that $\nu \perp F$ we have the following equality

$$\frac{\langle \nu(0), F(v)\rangle}{1 - \langle F(0), F(v)\rangle} = 2\frac{\langle F(v), \nu(0)\rangle - \langle F(0), \nu(0)\rangle}{|F(v) - F(0)|^2}.$$

Let us call

$$g(v) = \langle F(v), \nu(0)\rangle.$$

The Taylor expansion of the previous function around 0 is the following:

$$g(v) = g(0) + Dg(0)v + \frac{1}{2}\langle D^2g(0)v, v\rangle + o(|v|^2).$$

Now, let us consider a curve $\gamma : I \to \Sigma$ in the form $\gamma(t) = F(tv)$ for $t \in I$. With that in mind, we have

$$Dg(0)v = \frac{d}{dt}\left.\langle F(tv), \nu(0)\rangle\right|_{t=0} = \langle \gamma'(0), \nu(0)\rangle = 0$$

and

$$\langle D^2g(0)v, v\rangle = \langle \gamma''(0), \nu(0)\rangle = -\langle \gamma'(0), \nabla_{\gamma'(0)}\nu(0)\rangle = h(v, v)$$

where the second equality arises taking derivatives of $\gamma'(t) \perp \nu(tv)$

$$0 = \frac{d}{dt} \left\langle \gamma'(t), \nu(tv) \right\rangle\big|_{t=0} = \langle \gamma''(0), \nu(0) \rangle + \langle \gamma'(0), \nabla_{\gamma'(0)}\nu(0) \rangle.$$

Therefore, the Taylor expansion around 0 has the form

$$g(v) = g(0) + \frac{1}{2}h(v,v) + o(|v|^2).$$

Going back to the quotient of our supremum, we have

$$2\frac{g(v)-g(0)}{|F(v)-F(0)|^2} = \left[h\left(\frac{v}{|v|^2}, \frac{v}{|v|^2} \right) + o(1) \right] \frac{|v|^2}{|F(v)-F(0)|^2}.$$

Since $DF(0) = \mathrm{id}$, we conclude that

$$\limsup_{v \to 0} \frac{\langle \nu(0), F(v) \rangle}{1 - \langle F(0), F(v) \rangle} = \lambda$$

where $\lambda \geq 0$ is one of the principal curvatures of Σ at x, the other being $-\lambda$ as a result of Σ being minimal. Since Σ has no umbilic points, we have that in fact $\lambda > 0$. Additionally, the minimality of Σ also gives us that $|A(x)| = \sqrt{2}\lambda > 0$.

Taking all that into account, we have proven that

$$\limsup_{x \to y,\ x \neq y} \sqrt{2} \frac{|\langle \nu(x), F(y) \rangle|}{|A(x)|(1 - \langle F(x), F(y) \rangle)} = 1$$

and from there, the proposition follows. $\square$

Proposition 2.1 naturally splits the problem into two cases: when $\kappa = 1$ and when $\kappa > 1$. The first case can be dispatched easily, but the second one is more involved and entails the application of a maximum principle for degenerate elliptic operators due to Bony [4] (cf. theorem 2.2). A detailed discussion of both cases will be carried out in the following sections, and we make note here of an important similarity between them: the use of the following result of Lawson.

Theorem 2.1 (Lawson, 1969)**.** *The only embedded flat minimal torus in $\mathbb{S}^3$ is the Clifford torus, up to congruence.*

Here, by flat torus we mean a surface of genus 1 such that the Gaussian curvature vanishes everywhere. We postpone a discussion of this result of Lawson to section 3.2. The main takeaway for us at this point is that to prove Lawson's conjecture it suffices to show that any minimal torus embedded in $\mathbb{S}^3$ is flat, and this is indeed Brendle's approach in each case. More precisely, he makes use of the following easy corollary.

Corollary 2.1. *The only embedded minimal torus in $\mathbb{S}^3$ with parallel second fundamental form is the Clifford torus, up to congruence.*

Proof. If $\nabla^\Sigma h \equiv 0$ then one can easily check that $h(e_i, e_j)$ are locally constant for any frame. The Gauss equation says

$$K_\Sigma = 1 + ||h(e_1, e_2)||^2 + h(e_1, e_1)h(e_2, e_2) \tag{3}$$

so the fact that h is parallel implies that K_Σ is constant. Actually, more is true: by Gauss-Bonnet, the intrinsic curvature must vanish identically. Hence Σ is flat and we conclude by Theorem 2.1. □

2.1 The case $\kappa = 1$

As mentioned above this case is relatively simple in terms of length, but the approach as presented in [5] can seem opaque and unmotivated. As such we will begin by outlining the geometric idea behind Brendle's strategy. He considers the real-valued function $Z : \Sigma \times \Sigma \to \mathbb{R}$ given by

$$Z(x, y) = \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \tag{4}$$

where $\Psi(x) = \frac{1}{\sqrt{2}}|A(x)|$ – notice that, because our torus is minimal, this is the magnitude of the maximal principal curvature at x. This function has the following properties.

(i) For each $x \in \Sigma$, if $Z(x, \cdot) \geq 0$ then there exists a ball of boundary curvature $\Psi(x)$ contained inside Σ (that is, inside the region bounded by Σ) touching x.

(ii) For each $x \in \Sigma$, if $y \in \Sigma \setminus \{x\}$ is such that $Z(x, y) = 0$, then then there exists a ball of boundary curvature $\Psi(x)$ contained inside Σ touching both x and y. Moreover, this is the largest possible such ball.

(iii) For each $x \in \Sigma$, if $Z(x, \cdot) \equiv 0$ on some $U \subset \Sigma$ then U lies on the boundary of a ball of boundary curvature $\Psi(x)$ centred at $x - \nu(x)/\Psi(x)$.

The reason that (4) satisfies these is explored in §4.1.

Now, by Corollary 2.1 what we would like to show is that $\nabla^\Sigma h \equiv 0$. Thanks to the Codazzi equations and the minimality of our embedding, we have a lot of extra symmetries for the tensor $\nabla^\Sigma h$ which reduce this task to showing that $\nabla^\Sigma_{e_i} h(e_i, e_i) = 0$ for some frame $\{e_1, e_2\}$ – this is shown in the proof of the next proposition. To achieve this, Brendle's idea is to look at the function $f : t \mapsto Z(\bar{x}, \gamma(t))$ for fixed $\bar{x} \in \Sigma$ and γ the geodesic through $\bar{x}$ with initial velocity equal to the direction of maximum principal curvature. If we can show that f is locally constant (that is, locally identically 0) around $t = 0$, then by property (iii) above the curve γ is locally tangent to a sphere. If we have $\nabla^\Sigma h \equiv 0$, then this means that the maximum principal curvature of points along γ near $t = 0$ is constant. In this way, we can heuristically understand Brendle's strategy in the next proof as trying to reverse this line of reasoning.

Proposition 2.2. *If $\kappa = 1$, then $F : \Sigma \to \mathbb{S}^3$ is congruent to the Clifford torus.*

Proof. Let Ψ and Z be as above. The fact that $\kappa = 1$ implies that $Z(x, y) \geq 0$ for all $x, y \in \Sigma$. Let $\bar{x} \in \Sigma$ and let (e_1, e_2) be a basis of $T_{\bar{x}}\Sigma$ in which h is diagonal. In particular we can choose it that $h(e_1, e_1) = -\Psi(\bar{x})$ and $h(e_2, e_2) = \Psi(\bar{x})$. Finally let γ be the geodesic in Σ through $\bar{x}$ with initial velocity e_1 and consider

$$f(t) := Z(\bar{x}, \gamma(t)) = \Psi(\bar{x})(1 - \langle \bar{x}, \gamma(t)\rangle) + \langle \nu(\bar{x}), \gamma(t)\rangle$$

where we have identified Σ with $F(\Sigma)$. We compute

$$f'(t) = -\langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle$$

and using Lemma B.4 we get

$$f''(t) = \langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \gamma(t)\rangle - h(\gamma'(t), \gamma'(t))\langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle$$

whence it follows that

$$\begin{aligned} f'''(t) = {} & \langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \gamma'(t)\rangle \\ & - (\nabla^{\Sigma}_{\gamma'} h)(\gamma'(t), \gamma'(t))\langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \nu(\gamma(t))\rangle \\ & - h(\gamma'(t), \gamma'(t))\langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \nabla^0_{\gamma'}\nu(\gamma(t))\rangle. \end{aligned}$$

Now f is non-negative and $f(0) = f'(0) = f''(0) = 0$, so we must have that $f'''(0) = 0$, i.e.

$$(\nabla^{\Sigma}_{e_1} h)(e_1, e_1) - h(e_1, e_1)\langle \Psi(\bar{x})\bar{x} - \nu(\bar{x}), \nabla^0_{e_1}\nu(\bar{x})\rangle = 0$$

which implies that $(\nabla^{\Sigma} h)(e_1, e_1, e_1) = 0$ because $\nabla^0_{\gamma'}\nu$ is tangent to Σ. An identical argument yields $(\nabla^{\Sigma} h)(e_2, e_2, e_2) = 0$. By the Codazzi equations and the symmetry of h we also have

$$\nabla^{\Sigma} h(e_i, e_j, e_k) = \nabla^{\Sigma} h(e_k, e_j, e_i) \quad \text{and} \quad \nabla^{\Sigma} h(e_i, e_j, e_k) = \nabla^{\Sigma} h(e_j, e_i, e_k)$$

for $i, j, k = 1, 2$. Putting all of this together one readily sees that $\nabla^{\Sigma} h \equiv 0$. $\square$

2.2 The case $\kappa > 1$

The second case in Brendle's proof is substantially more involved. It again centres on the quantity κ and the functions Z and Ψ, but this time the technique of Prop. 2.2 fails because things no longer cancel nicely. Instead the analysis of Z is more subtle, and relies on PDE theory. The first step consists on finding a **Simons-type identity** for the function Ψ. The derivation of this identity follows quickly from Theorem 5.3.1 in [17] and the non-umbilic property of minimal tori proved in 3.2, and is carried out in [5].

Proposition 2.3. *Let $F : \Sigma \to \mathbb{S}^3$ be an embedded minimal torus in $\mathbb{S}^3$. Then $\Psi = \frac{1}{\sqrt{2}}|A|$ satisfies the partial differential equation*

$$\Delta_{\Sigma}\Psi - \frac{|\nabla_{\Sigma}\Psi|^2}{\Psi} + (|A|^2 - 2)\Psi = 0.$$

Of course, in this context $Z(x,y)$ looks like

$$Z(x,y) = \kappa\Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle.$$

because $\kappa > 1$. This is still a non-negative function and Proposition 2.1 implies that there exist $(\bar{x}, \bar{y}) \in \Sigma \times \Sigma$ such that $\bar{x} \neq \bar{y}$ and $Z(\bar{x}, \bar{y}) = 0$.

Now, probably the most interesting part of Brendle's proof is what comes next: his use of the **strict maximum principle for degenerate elliptic operators**. We state this result here, but will postpone discussion of it until secton 3.3.

Theorem 2.2 (Bony-Brendle)**.** *Let Ω be an open subset of a Riemannian manifold and let $X_1, \ldots, X_m$ be smooth vector fields on Ω. Let $\varphi : \Omega \to \mathbb{R}$ be a smooth non-negative function satisfying*

$$\sum_{j=1}^{m} (D^2\varphi)(X_j, X_j) \leq -\inf_{|\xi|\leq 1} (D^2\varphi)(\xi, \xi) + L|d\varphi| + L\varphi \tag{5}$$

for some $L > 0$. Let $F = \{\varphi = 0\}$ and let $\gamma : [0,1] \to \Omega$ be a smooth curve with $\gamma(0) \in F$. If the velocity of γ lies in the span of the X_i, for $i = 1, ..., m$ and for all time, then γ lies in F for all time.

To see how this maximum principle appears in Brendle's proof, we introduce the following set

$$\Omega = \{\bar{x} \in \Sigma \,|\, \text{there exists } \bar{y} \in \Sigma \setminus \{\bar{x}\} \text{ with } Z(\bar{x}, \bar{y}) = 0\},$$

which, as noted above, is non-empty. From here, the argument proceeds via the following steps:

(i) Show that $\nabla\Psi = 0$ on Ω, so in particular $\Delta_\Sigma \Psi = 0$ and $\Psi \equiv 1$ on Ω by proposition 2.3.

(ii) Use the maximum principle to show that Ω is open.

(iii) Apply unique continuation for solutions to elliptic PDEs to conclude that $\Psi \equiv 1$ on all of Σ, so in particular $|A|$ is constant.

(iv) Note that this implies that Σ is flat and conclude via Lawson's theorem.

The proof of the first two of these follow from the estimate

$$(**) \qquad \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(x,y)$$
$$\leq -\frac{\kappa^2 - 1}{\kappa} \frac{\Psi(x)}{1 - \langle F(x), F(y)\rangle} \sum_{i=1}^{2} \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle^2$$
$$+ \tilde{\Lambda}(x,y) \left(Z(x,y) + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial x_i}(x,y) \right| + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial y_i}(x,y) \right| \right)$$

for some continuous $\tilde{\Lambda} : \Sigma \times \Sigma \setminus \Delta \to \mathbb{R}^+$ and all $x \neq y$ in Σ. A derivation of this estimate is carried out in Appendix C of the present work; in [5] the details are omitted.

Proposition 2.4 (Step (i))**.** *We have $\nabla \Psi = 0$ on Ω.*

Proof. Let $\bar{x} \in \Omega$ with $\bar{y} \neq \bar{x}$ being the corresponding point. Then Z attains a minimum of 0 at $(\bar{x}, \bar{y})$, which means that its second derivatives should be positive while its first derivatives should vanish. Hence $(**)$ implies that

$$\begin{aligned} 0 \leq \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(x,y) \\ \leq -\frac{\kappa^2 - 1}{\kappa} \frac{\Psi(x)}{1 - \langle F(x), F(y) \rangle} \sum_{i=1}^{2} \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle^2 \leq 0. \end{aligned}$$

Now $\kappa > 1$ by assumption, so we must have that

$$\left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle = 0$$

for each i. Recalling that

$$0 = \frac{\partial Z}{\partial x_i}(\bar{x}, \bar{y}) = \kappa \frac{\partial \Psi}{\partial x_i}(x)(1 - \langle F(x), F(y) \rangle) - \kappa(\Psi(x) - h_i^i(x)) \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle$$

gives the result. $\square$

Proposition 2.5 (Step (ii))**.** *The set Ω is open.*

Proof. Let $\bar{x} \in \Omega$ with corresponding $\bar{y} \neq \bar{x}$ in Σ as before. Fix a small precompact neighbourhood U of $(\bar{x}, \bar{y})$ in $\Sigma \times \Sigma \setminus \Delta$ and let $L = ||\tilde{\Lambda}||_{L^\infty(U)} < \infty$. Then by $(**)$ we have

$$\begin{aligned} \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(x,y) \\ \leq L \left(Z(x,y) + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial x_i}(x,y) \right| + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial y_i}(x,y) \right| \right) \end{aligned}$$

for every $(x, y) \in U$, because the first term is always non-positive. If we define the local vector fields

$$\xi_1 := \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} \quad \text{and} \quad \xi_2 := \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}$$

we may rewrite the above inequality as

$$\sum_{j=1}^{2} (D^2 Z)(\xi_j, \xi_j) \leq L \left(Z(x,y) + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial x_i}(x,y) \right| + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial y_i}(x,y) \right| \right).$$

Then by the maximum principle of Bony-Brendle (Theorem 2.2) we get that if $\gamma : [0,1] \to U$ is a smooth curve with $\gamma(0) \in \{Z = 0\}$ and $\gamma'(t) = f_1(t)\xi_1(t) + f_2(t)\xi_2(t)$ for some smooth $f_1, f_2 : [0,1] \to \mathbb{R}$ and all $t \in [0,1]$, then $\gamma(t) \in \{Z = 0\}$ for all $t \in [0,1]$. We now note that the distribution generated by the ξ_i is involutive, and we let V denote the integral submanifold of this distribution through $(\bar{x}, \bar{y})$. The key point is that $V \subset \{Z = 0\} \cap U$. Indeed, if $(x, y) \in V$ then we may connect (x, y) to $(\bar{x}, \bar{y})$ using a curve in V and conclude via the above.

With this fact in hand, we are essentially done. The submanifold V sits non-vertically over the first copy of Σ, and so when projected to this copy it will cover an open neighbourhood of $\bar{x}$. By construction, this neighbourhood will be contained inside Ω. $\square$

To proceed from here we use a formulation of the unique continuation principle due to Aronszajn in [3].

Theorem 2.3 (Unique continuation)**.** *Let L be a linear elliptic operator of second order and $C > 0$. Let $u : M \to \mathbb{R}$ be a C^2 function on a connected smooth manifold M satisfying*

$$|Lu|^2 \leq C(|\nabla u|^2 + |u|^2).$$

Then if u vanishes on an open set in M, it is identically 0.

Proposition 2.6 (Step (iii))**.** *We have $\Psi \equiv 1$ on Σ.*

Proof. By Prop. 2.3 we have

$$\Psi \Delta_\Sigma \Psi = |\nabla \Psi|^2 - (|A|^2 - 2)|\Psi|^2 \tag{6}$$

the equality being satisfied in the whole torus Σ. Since $|\Psi|$ is bounded away from 0, the operator $L := \Psi \Delta_\Sigma$ is uniformly elliptic. If we now define $u = \Psi - 1$ then (6) gives us

$$Lu = |\nabla u|^2 - (|A|^2 - 2)(u+1)^2.$$

Using now that $|A|^2 = 2\Psi^2$ we get

$$\begin{aligned} Lu &= |\nabla u|^2 - 2(\Psi^2 - 1)(u+1)^2 \\ &= |\nabla u|^2 - 2(\Psi + 1)(\Psi - 1)(u+1)^2 \\ &= |\nabla u|^2 - 2(\Psi + 1)\Psi^2 u. \end{aligned}$$

Taking squares, we have

$$\begin{aligned} |Lu|^2 &\leq |\nabla u|^4 + 4(\Psi+1)^2\Psi^4|u|^2 + 4(\Psi+1)\Psi^2 u|\nabla u|^2 \\ &\leq \left(|\nabla u|^2 + 4(\Psi+1)\Psi^2 u\right)|\nabla u|^2 + 4(\Psi+1)^2\Psi^4|u|^2. \end{aligned}$$

Since Σ is a compact manifold, we can consider

$$C = \max\left\{\max_{x\in\Sigma}(|\nabla u|^2 + 4(\Psi+1)\Psi^2 u), \max_{x\in\Sigma}(4(\Psi+1)^2\Psi^4)\right\}$$

and thus get the estimate we wanted in Σ

$$|Lu|^2 \leq C(|\nabla u|^2 + |u|^2).$$

With that in mind, we apply theorem 2.3 and conclude that the function u, which vanishes on the open set $\Omega \subset \Sigma$, actually vanishes everywhere, and thus we conclude that $\Psi \equiv 1$ on Σ as we wanted. $\square$

Finally, to prove step (iv), we only need to realise that $|A| \equiv \sqrt{2}$ on Σ implies that the principal curvatures are ± 1. In particular, by the Gauss equation again (equation 3) we get that the Gauss curvature of Σ is constantly 0 and we can apply Lawson's theorem 2.1 to conclude that our embedded torus is the Clifford torus.

3 Principal results used in Brendle's proof

Let us now give an exposition on some of the key theorems that Brendle uses on his proof, which so far we have just stated and postponed further remarks. Here we will give some of the proofs for those (the ones that we consider of some interest for our work) and the geometric intuitions that we found compelling related to them.

3.1 No umbilic points

In this subsection, we give a proof of the crucial fact that any minimal torus in $\mathbb{S}^3$ has no umbilic points. This is necessary for the central quantity κ of Brendle's proof to even be well-defined. In the name of completeness, we also include a proof of Almgren's result on minimal 2-spheres in $\mathbb{S}^3$. We begin with some elementary observations about the totally geodesic case.

Lemma 3.1. *If $\Sigma \subset \mathbb{S}^3$ is a totally geodesic submanifold of codimension 1, then Σ has genus 0.*

Proof. The Gauss equation says $K_\Sigma = 1 + ||h(e_1, e_2)||^2 + h(e_1, e_1)h(e_2, e_2)$ where K_Σ is the intrinsic Gauss curvature of Σ. Being totally geodesic means that $h \equiv 0$, so $K_\Sigma \equiv 1$. Hence $g(\Sigma) = 0$ by Gauss-Bonnet. $\square$

Corollary 3.1. *The only totally geodesic submanifolds in $\mathbb{S}^3$ of codimension 1 are the equators.*

Proof. The geodesics of $\mathbb{S}^3$ are circles of maximal radius, so this follows immediately from the previous lemma. $\square$

Theorem 3.1 (Almgren)**.** *Let $F : \mathbb{S}^2 \to \mathbb{S}^3$ be a minimal immersion. Then $F(\mathbb{S}^2)$ is an equator.*

Proof. We identify $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$ via stereographic projection and denote by $z = x + iy$ the complex coordinate on $\mathbb{C}$. Consider $f := h(\partial_z, \partial_z)$ as a map $\mathbb{S}^2 \setminus \{\infty\} \to \mathbb{C}$. We have

$$\begin{aligned} f = h(\partial_z, \partial_z) &= h(\partial_x, \partial_x) - 2ih(\partial_x, \partial_y) - h(\partial_y, \partial_y) \\ &= h_1^1 - 2ih_1^2 - h_2^2. \end{aligned}$$

Then

$$\begin{aligned} \bar{\partial} f &= \partial_x(h_1^1 - 2ih_1^2 - h_2^2) + i\partial_y(h_1^1 - 2ih_1^2 - h_2^2) \\ &= (\partial_x h_1^1 - \partial_x h_2^2 + 2\partial_y h_1^2) + i(\partial_y h_1^1 - \partial_y h_2^2 - 2\partial_x h_1^2) \\ &= (2\partial_x h_1^1 + 2\partial_y h_1^2) + i(-2\partial_y h_2^2 - 2\partial_x h_1^2) \\ &= 0 \end{aligned}$$

where we used the minimality of F for the third equality and (15) for the fourth. Thus f is holomorphic. We also have that $f \to 0$ as $p \to \infty$, since $||\partial_z(p)||_{\mathbb{S}^2} \to 0$ as can be checked explicitly using the definition of stereographic projection. This means that f extends continuously by 0 over the point at infinity, and standard properties of holomorphic continuation imply that this extension is moreover holomorphic. Liouville's theorem then implies that f is identically 0, so in particular

$$h(\partial_x, \partial_x) = h(\partial_y, \partial_y) \quad \text{and} \quad h(\partial_x, \partial_y) = 0$$

on all of $\mathbb{S}^2$. But F is minimal, so the first identity yields that both $h(\partial_x, \partial_x)$ and $h(\partial_y, \partial_y)$ vanish identically and hence $h \equiv 0$, i.e. F immerses $\mathbb{S}^2$ as a totally geodesic submanifold of $\mathbb{S}^3$. We conclude by Corollary 3.1. □

Theorem 3.2 (Lawson)**.** *Let Σ be a torus and let $F : \Sigma \to \mathbb{S}^3$ be a minimal immersion. Then $F(\Sigma)$ has no umbilic points.*

Proof. The argument is familiar: by uniformisation, we may think of Σ as $\mathbb{C}/\Gamma$ for a lattice $\Gamma \subset \mathbb{C}$. Then using the natural complex coordinate z we can again consider the function $h(\partial_z, \partial_z)$. By Liouville, this is equal to a constant c. If $c = 0$, then $F(\Sigma)$ is totally geodesic, which contradicts the genus being 1. Hence $c \neq 0$ and so $|h|$ never vanishes. □

3.2 Lawson's rigidity theorems

In all, the proof by Brendle of Lawson's conjecture is based on showing that the minimal torus inside $\mathbb{S}^3$ always has vanishing Gaussian curvature. That way, he is able to invoke a classical result by Lawson in [15] (c.f. Theorem 2.1) and conclude that this minimal torus is congruent to the Clifford torus. This subsection is dedicated to elaborate on that key result by Lawson.

In appendix B, we show that the Laplacian of a minimal immersion into the sphere $\mathbb{S}^3$ has to verify a nice relation (cf. equation 16), that is often useful for computations. Here, we will use a more general version of that expression, which can be obtained using the same arguments that were shown in the proof of the corresponding lemma.

Theorem 3.3. *An isometric immersion $\Psi : M \to \mathbb{S}^{n+1}$, where M is a p-dimensional manifold, is a minimal immersion if and only if*

$$\Delta_M \Psi = -p\Psi$$

where Δ_M is the Laplace-Beltrami on M.

Moreover, it is also important to know that the product of spheres

$$\mathbb{S}^p\left(\sqrt{\frac{p}{n}}\right) \times \mathbb{S}^q\left(\sqrt{\frac{q}{n}}\right) \subset \mathbb{S}^{n+1},$$

defined as

$$\left\{(x_1, ..., x_{p+1}, y_1, ..y_{q+1}) \in \mathbb{R}^{n+2} : \sum_{i=1}^{p+1} x_i^2 = \frac{p}{n},\ \sum_{i=1}^{q+1} y_i^2 = \frac{q}{n}\right\}$$

where $q + p = n$ and $p \geq 0$, is a minimal surface of the sphere $\mathbb{S}^{n+1}$.

The precise rigidity theorems that Lawson proved in [15] are stated in quite a more general setting than the one which we have been dealing with so far. Let M be an n-dimensional smooth manifold and $\Psi : M \to \mathbb{S}^{n+1}$ an isometric minimal immersion.

Theorem 3.4 (Lawson, 1969)**.** *If the scalar curvature (which here is defined as the average of the sectional curvatures at the point) of M is identically equal to $(n-2)/(n-1)$, then, up to rotations of $\mathbb{S}^{n+1}$, $\Psi(M)$ is an open submanifold of one of the minimal products*

$$\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$$

for $k = 1, ..., \left[\frac{n}{2}\right]$.

Theorem 3.5 (Lawson, 1969)**.** *If the Ricci curvature of M is parallel, then, up to rotations of $\mathbb{S}^{n+1}$, $\Psi(M)$ is an open submanifold of one of the minimal products*

$$\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$$

for $k = 0, ..., \left[\frac{n}{2}\right]$.

For surfaces, there is only one independent component of the curvature tensor, so that the Ricci curvature and the scalar curvature are the same. Moreover, in this context the sectional curvature and the Gauss curvature are also the same. Using Gauss equation and the previous theorems, we can conclude the following, which actually provides more information than we need.

Corollary 3.2. *If Σ is a minimal surface in $\mathbb{S}^3$ of constant Gaussian curvature K, then*

- *either $K = 1$ and Σ is totally geodesic,*
- *or $K = 0$ and Σ is an open piece of the Clifford torus.*

Now, let us discuss the strategy followed by Lawson in [15] to prove the previous theorems. The proof is divided into the two following propositions. Let $A \in \Gamma(\mathrm{End}(TM))$ denote the shape operator of M.

Proposition 3.1. *If A is parallel over M, then, $\Psi(M)$ is an open submanifold of one of the minimal products*

$$\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$$

for $k = 0, ..., \left[\frac{n}{2}\right]$.

Proposition 3.2. *If $\Psi : M \to \mathbb{S}^{n+1}$ is a minimal immersion and either*

1. *the scalar product is constantly*
$$\kappa = \frac{n-2}{n-1}$$
or,
2. *the Ricci curvature is parallel,*

then A is parallel.

Let us begin discussing the **proof of proposition 3.1**. Consider $p \in M$ and let A_p be the shape operator acting on T_pM. Let $\{e_1, ..., e_k\}$ be a basis of eigenvectors of A_p and $\{\lambda_1, ..., \lambda_k\}$ be the corresponding eigenvalues. Note that since A_p is symmetric, the eigenvectors associated to different eigenvalues will be orthogonal.

The first thing to notice is that A has the same eigenvalues with the same multiplicities at all points of M. Let $q \in M$ be another point and consider a path joining p and q. We define vector fields ξ_i by parallel transport of the vectors e_i, for $i = 1, ..., k$, along that path. Since A is parallel, we obtain the the equation

$$A\xi_i = \lambda_i \xi_i$$

must be satisfied at q as well. This fact leaves us with two cases.

First, if $A_p = 0$, then in fact we have that $A \equiv 0$ and that the manifold M is totally geodesic. Using Gauss equation we see that this is equivalent to the Gauss curvature being equal than 1.

The case $A_p \neq 0$ is a bit more involved. The first idea worth noting is that, if $\mathcal{H}_p$ denotes the holonomy group of M at p, we have that

$$HA_p = A_pH, \qquad \forall\ H \in \mathcal{H}_p.$$

This can be seen by considering the vector fields $\xi_i = \xi_i(t)$ obtained by parallel transport of e_i, $i = 1, ..., k$, along the loop generating the holonomy transformation H. As before, we have that each of these vectors are eigenvectors of the corresponding shape operator, with eigenvalue λ_i. Since $H(e_i) = \xi_i(1)$, we have

$$A_p H(e_i) = \lambda_i H(e_i) = H(\lambda_i e_i) = H A_p(e_i)$$

for every $i = 1, ..., k$ as we wanted. This fact implies that, if E_i is an eigenspace of A_p in T_pM, then we get

$$H(E_i) \subset E_i.$$

Now, the Ambrose-Singer theorem tells us that the lie algebra of $\mathcal{H}_p$ is generated by elements of the form

$$\tau^{-1} \circ R(u, v) \circ \tau$$

where τ is the parallel transport along a loop based at p and R is the Riemann curvature tensor, $u, v \in T_pM$. Taking $\tau = \mathrm{id}$ we get that the Riemann curvature gives us a holonomy transformation. Using what we have just seen, for e_i and e_j with $\lambda_i \neq \lambda_j$, we have that the corresponding sectional curvature vanishes

$$\langle R(e_i, e_j)e_j, e_i \rangle = 0.$$

Using Gauss equation we realise that

$$0 = \langle R(e_i, e_j)e_j, e_i \rangle = 1 + \lambda_i \lambda_j.$$

This last equality implies that, if $A_p \neq 0$, then we can have at most two different eigenvalues. Using also that the trace of A vanishes we get that these eigenvalues are

$$\lambda_1 = \pm\sqrt{\frac{n_2}{n_1}}, \qquad \lambda_2 = \mp\sqrt{\frac{n_1}{n_2}}$$

where $n_1 + n_2 = n$ and λ_k has multiplicity $n_k \geq 1$, $k = 1, 2$.

Now, it is not hard to check that the subbundles

$$\mathcal{D}_k \{x \in T_pM : p \in M, Ax = \lambda_k x\}$$

for $k = 1, 2$ are in fact involutive distributions. Therefore, we can apply Frobenius Theorem and obtain a local chart $(U, (x_1, ..., x_{n_1}, y_1, .., y_{n_2}))$ around p such that p is mapped to the origin and the level sets

$$E(y') = \{(x, y) \in U : y_1 = y'_1, ..., y_{n_2} = y'_{n_2}\},$$

$$F(x') = \{(x, y) \in U : x_1 = x'_1, ..., x_{n_1} = x'_{n_1}\},$$

are the integral leaves of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Moreover, one can show that U is isometric to the product

$$E(0) \times F(0).$$

Since the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are invariant under parallel transport, we have that their integral leaves are totally geodesic. Thus, from Gauss equation we get that the manifolds $E(0)$ and $F(0)$ have constant curvature equal to $\frac{n}{n_1}$ and $\frac{n}{n_2}$ respectively. Therefore, after possibly reducing U, there is an isometric embedding

$$\Phi : U \to \mathbb{S}^{n_1}\left(\sqrt{\frac{n_1}{n}}\right) \times \mathbb{S}^{n_2}\left(\sqrt{\frac{n_2}{n}}\right).$$

Now, let M^* be the universal cover of M with covering map $\pi : M^* \to M$. Fix some $p^* \in \pi^{-1}(p)$ and consider U^* the unique lift of U containing p^*. Using standard continuation techniques in simply connected manifolds, we can lift the previous isometry to a local isometry defined on the whole manifold M^*

$$\Phi' : M^* \to \mathbb{S}^{n_1}\left(\sqrt{\frac{n_1}{n}}\right) \times \mathbb{S}^{n_2}\left(\sqrt{\frac{n_2}{n}}\right)$$

such that $\Phi'|_{U^*} = \Phi \circ \pi|_{U^*}$.

Finally, we can lift the immersion Ψ to an immersion $\Psi^* : M^* \to \mathbb{S}^{n+1}$ and considering the minimal embedding

$$i : \mathbb{S}^{n_1}\left(\sqrt{\frac{n_1}{n}}\right) \times \mathbb{S}^{n_2}\left(\sqrt{\frac{n_2}{n}}\right) \to \mathbb{S}^{n+1}$$

we can define $\Phi^* = i \circ \Phi'$. The last step of this proof consists on checking that $\Psi^* = \Phi^*$ on M^*. For that, Lawson uses uniqueness of solutions of differential equations. Since the precise computations do not help understand any geometric features of the problem, we will omit them here.

With this last step we have $\Psi \circ \pi = \Phi^*$, and therefore, Ψ immerses M into one of the desired product of spheres. This concludes the proof of proposition 3.1.

On the other hand, the **proof of proposition 3.2** involves taking the pertinent computations to show that, under the assumptions of the proposition, the shape operator is parallel. Let us review how those computations look like.

- First, we consider the first assumption, namely that the scalar curvature of our surface κ is constantly equal to $\frac{n-2}{n-1}$.

  Taking traces in the Gauss Equation twice we encounter the following quality

$$\kappa = 1 - \frac{|A|^2}{n(n-1)}$$

so that, in our situation, we conclude that $|A|^2 \equiv n$.

Using Simon's fundamental equation (cf. theorem 5.3.1 in [17]) we get

$$\Delta A = nA - |A|^2 A = 0.$$

With that in mind, we have that

$$|\nabla A|^2 = -\langle \Delta A, A\rangle = 0,$$

and we conclude that the shape operator is parallel, as we wanted.

- Now, let us assume that the Ricci tensor is parallel. Taking the trace of the Gausss equation, only once in this case, we get

  $$\mathrm{Ric}(u, v) = -(n-1) + \langle A(u), A(v)\rangle = -(n-1) + \langle A^2(u), v\rangle$$

  so that we conclude that A^2 is parallel. Therefore, the eigenvalues of A^2 (and thus the eigenvalues of A) are constant in value and multiplicity over the manifold M. Let $\{\lambda_1, ..., \lambda_k\}$ be those eigenvalues and let us define

  $$\mathcal{D}_{\lambda_j}(p) := \{u \in T_pM : A_p(u) = \lambda_j u\}$$

  for every $p \in M$ and $k = 1, ..., k$.

  First notice that, from A^2 being parallel we obtain that, for $u \in T_pM$, we have

  $$(\nabla_u A) \circ A + A \circ (\nabla_u A) = 0$$

  in T_pM.

  Now, let us check that for $u \in D_{\lambda_i}(p)$ and $v \in D_{\lambda_j}(p)$ with $1 \leq i, j \leq k$ we have

  $$(\nabla_u A)^2(v) = 0.$$

  From the previous equality, we have

  $$A(\nabla_u A(v)) = -\nabla_u A(A(v)) = -\lambda_j \nabla_u A(v)$$

  so that $\nabla_u A(v) \in \mathcal{D}_{-\lambda_j}(p)$.

  - Let $\lambda_i \neq \lambda_j$. From the Mainardi-Codazzi equation

    $$\nabla_u A(v) = \nabla_v A(u)$$

    and thus

    $$\nabla_u A(v) \in \mathcal{D}_{-\lambda_j}(p) \cap \mathcal{D}_{-\lambda_i}(p)$$

    so that we conclude $\nabla_u A(v) = 0$.

– If $\lambda_i = \lambda_j \neq 0$ then $\nabla_u A(v) \in \mathcal{D}_{-\lambda_i}(p)$ and by the same argument, since $\lambda_i \neq -\lambda_i$ we also get $\nabla_u A(v) = 0$.

– Finally, assume that $\lambda_i = \lambda_j = 0$. Extend v to a vector field Y by parallel transport along geodesics passing through p. Then we have $A^2 Y \equiv 0$ and, since A is symmetric also $AY = 0$. Therefore, we find

$$\nabla_u A(v) = \nabla_u (AY) - A(\nabla_x Y) = 0.$$

With this in mind, considering $\{e_i : 1 \leq i \leq n\}$ an orthonormal basis of eigenvectors of A for T_pM we get

$$\langle \nabla A, \nabla A \rangle = \sum_{i,j=1}^{n} \langle (\nabla_{e_i} A)(e_j), (\nabla_{e_j} A)(e_i) \rangle = \sum_{i,j=1}^{n} \langle (\nabla_{e_i} A)^2 (e_j), e_j \rangle = 0$$

and we are done.

3.3 Bony's maximum principle

The proof of Lawson's Conjecture in the case $\kappa > 1$ relies heavily on theorem 2.2. This theorem is a generalisation stated by Brendle in [6] of a maximum principle introduced by Bony [4] for degenerate second order differential operators. Let us explain in this subsection the geometric intuition behind this theorem and its relation with classical maximum principles.

Let us first recall the statement of the **classical maximum principle**. We will follow the standard reference [10], in particular its section 6.4. This theorem is stated for (uniformly) elliptic operators in $\Omega \subset \mathbb{R}^n$ an open bounded set. These are second order operators of the form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u$$

where the second order coefficients are symmetric, that is, $a_{ij} = a_{ji}$ for every $i, j = 1, ..., n$, and there exists some constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. Observe that this condition means that the symmetric $n \times n$ matrix

$$A(x) = (a_{ij}(x))$$

is positive definite, thus non-degenerate, with smallest eigenvalue greater than or equal to α for a.e. $x \in \Omega$.

For the statement of the classical maximum principle we additionally need to assume that the functions a_{ij}, b_i, c are continuous in Ω.

Theorem 3.6 (Strong maximum principle with $c \geq 0$.). *Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \geq 0$ in Ω. Suppose also that Ω is connected.*

- *If $Lu \leq 0$ in Ω and u attains a non-negative maximum over $\overline{\Omega}$ at an interior point then u is constant within Ω.*
- *If $Lu \geq 0$ in Ω and u attains a non-positive minimum over $\overline{\Omega}$ at an interior point then u is constant within Ω.*

In fact, we are mostly interested in the second bullet point of the previous theorem, since **Bony-Brendle's theorem** 2.2 concerns the minima of the function involved in the statement. Let us recall now the statement of that theorem. We have a set $X_1, \dots, X_m$ of smooth vector fields on Ω and a smooth non-negative function $u : \Omega \to \mathbb{R}$ satisfying

$$\sum_{j=1}^{m} (D^2u)(X_j, X_j) \leq -C \inf_{|\xi| \leq 1} (D^2u)(\xi, \xi) + C|du| + Cu$$

for some $C > 0$. Then, Bony-Brendle's theorem states that, if $F = \{u = 0\}$ and $\gamma : [0,1] \to \Omega$ is a smooth curve with $\gamma(0) \in F$ and

$$\gamma'(t) \in \operatorname{span}(X_1, ..., X_m), \quad \forall t \in [0,1];$$

then $\gamma(t) \in F$ for all $t \in [0,1]$.

At first sight, this statement might not seem closely related to the classical maximum principle that we have introduced. **Two questions arise** in this sense: how the assumptions of the classical maximum principle are related to the hypothesis of Bony-Brendle's theorem and how the statement of Bony-Brendle's theorem might be interpreted as a maximum principle.

As a first approach to both these questions, as we said before, the statement of Bony-Brendle's theorem can be actually understood as a minimum principle. We have a non-negative function $u : \Omega \to \mathbb{R}$ satisfying some conditions, and Bony-Brendle's theorem gives us information about the set $F = \{u = 0\}$, which in this case is just the set of minima of the function u. In particular, the theorem tells us that the integral manifolds described by the distribution of fields $X_1, ..., X_m$ stay inside of this set.

Now, concerning specifically the first of those questions, consider a non-negative function $u : \Omega \to \mathbb{R}$ in the assumptions of the classical minimum principle, that is, such that $Lu \geq 0$ and attaining a non-positive minimum over $\overline{\Omega}$. In this case having a non-positive minimum means that u vanishes at the minimum, which is also implied on the assumptions of the Bony-Brendle Theorem when they assume that $F = \{u = 0\}$ is non-empty.

Let $X_1, ..., X_n$ be a collection of smooth vector fields such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n} a_{ij}(x)(D^2u)(\partial_i, \partial_j) = \sum_{k=1}^{n} (D^2u)(X_k, X_k) \tag{7}$$

up to first order. The existence of such a collection is justified in proposition B.1 and uses the Gram-Schmidt process. We therefore have

$$Lu = -\sum_{k=1}^{n} D^2u(X_k, X_k) + \langle B, \nabla u\rangle + cu \geq 0$$

where B is a continuous vector field incorporating both the b_i from our initial expression for L and the additional first order terms picked up from (7). Now continuity and the fact that Ω is bounded implies

$$\sum_{k=1}^{n}(D^2u)(X_k, X_k) \leq C|du| + Cu \tag{8}$$

for some $C > 0$, which is precisely the bound that appears in the statement of Bony-Brendle's theorem. The term

$$-C \inf_{|\xi|\leq 1}(D^2u)(\xi,\xi)$$

is always non-negative, so it can always be added to the right-hand side of the bound 8. To check this, we note that at every point the hessian verifies one of the following conditions: it has no negative eigenvalues or it has at least one negative eigenvalue. In the former case, which happens when we are at a minimum of the function u, the matrix is positive semi-definite, which means that

$$(D^2u)(\xi,\xi) \geq 0, \quad \forall \xi \in \mathbb{R}^n,$$

and since for $\xi = 0$ we have that $(D^2u)(\xi,\xi) = 0$ we can conclude that the infimum is precisely 0. On the other hand, let us assume that the hessian at some point has a negative eigenvalue. In that case, the value of the infimum equals the value of the most negative eigenvalue and therefore the quantity

$$-C \inf_{|\xi|\leq 1}(D^2u)(\xi,\xi)$$

is strictly positive. With this, we conclude that indeed the assumptions of the classical maximum principle imply the assumptions of the Bony-Brendle maximum principle.

Now, let us address the second question that concerned us about these theorems, which is how to interpret the Bony-Brendle theorem as a minimum principle. In the previous situation, where we assumed the hypothesis of the classical maximum principle, we had that u vanished over Ω. Therefore, the conclusion of Bony-Brendle's Theorem was also satisfied trivially: $F = \Omega$ so that every curve in Ω is trivially contained in F. Moreover, since the vector fields $X_1, ..., X_n$ from proposition B.1 form a basis on $\mathbb{R}^n$, we can write any vector representing the velocity of a certain curve as linear combination of those. In that sense, we begin to see that the statement of Bony and Brendle is weaker than the classical maximum principle. However, we can go further in the geometric meaning of this generalisation of the maximum principle.

As we have seen for the case of the classical maximum principle, we could find a basis of vector fields that verified the bound on the Bony-Brendle's Theorem. But this theorem by itself never assumes that the vector fields satisfying the bound must form a basis of the tangent space. What is more: it never assumes that there are $n = \dim \Omega$ of these fields, but rather some number $m \leq n$. These vector fields satisfying the bound are the ones that will describe the integral manifold that stays inside the set of minima of the function. The geometric intuition here is that, when we go to the realm of degenerate differential operators, we lose some of the directions in Ω in which, for elliptic operators, we would have stayed inside of the set of minima.

Additionally, observe that the bound for elliptic operators is lower that the general bound that Bony-Brendle's theorem establishes for degenerate operators. This also makes sense if we think that, when turning to degenerate operators, we lose control over the hessian acting on these vector fields.

4 Where does this come from?

One of the most natural questions which arises when investigating Brendle's proof is concerned with the origin of his function Z and the quantity κ, and in particular with whether they contain intuitive geometric content. In this section, we try and address this by explaining the definition in light of the work of Andrews on the mean curvature flow.

4.1 Non-collapsing properties of the mean curvature flow

In [1], Andrews gives a short direct proof of the "δ-non-collapsing property" for mean-convex solutions to the mean curvature flow. Recall that a 1-parameter family M_t of hypersurfaces in some ambient manifold **evolves by the mean curvature flow** if

$$\frac{\partial}{\partial t} M_t = -H_t \nu_t \tag{9}$$

where H_t denotes the mean curvature of M_t and ν_t is a unit normal. A solution to (9) is called **mean-convex** if $H_t > 0$ everywhere for all time. Finally, we say that a closed hypersurface M bounding a region Ω is **δ-non-collapsing** for $\delta > 0$ if for every $x \in M$ there exists an open ball B of radius $\delta / H(x)$ contained in Ω with $x \in \partial B$. The main result of [1] is the following, which was first proved using more involved methods by Sheng and Wang in [19].

Theorem 4.1 (Sheng-Wang, Andrews)**.** *Let M_t be a smooth family of embedded closed hypersurfaces in $\mathbb{R}^{n+1}$ evolving by mean curvature flow for $t \in [0, T)$. Assume that M_t is mean-convex for every $t \in [0, T)$. Then if M_0 is δ-non-collapsed, it follows that M_t is δ-non-collapsed for $0 < t < T$.*

The key to Andrews' argument is the following observation. For a given point x on a hypersurface $M \subset \mathbb{R}^{n+1}$ which bounds a region Ω, let $B \subset \Omega$ be an open ball such that

$x \in \partial B$. Denoting the outward unit normal to M by ν, it is clear that $x - r\nu(x)$ is the centre of B for some $r > 0$. The condition that $B \subset \Omega$ implies that no other point of M is within a distance r of this centre, i.e.

$$\begin{aligned} B \subset \Omega &\iff |y - x + r\nu(x)|^2 \geq r^2 \text{ for all } y \in M \\ &\iff |y-x|^2 + 2r\langle \nu(x), y - x\rangle + r^2 \geq r^2 \text{ for all } y \in M \\ &\iff \frac{1}{r} \geq \frac{2\langle \nu(x), x-y\rangle}{|y-x|^2} \text{ for all } y \in M \\ &\iff \sup_{x \neq y} \frac{2\langle \nu(x), x-y\rangle}{|y-x|^2} \leq \frac{1}{r}. \end{aligned}$$

If we now define

$$k(x) = \sup_{y \neq x} \frac{2\langle \nu(x), x-y\rangle}{|y-x|^2}$$

then it becomes clear that M is δ-non-collapsed if and only if

$$\max_M \frac{k}{H} \leq \frac{1}{\delta}. \tag{10}$$

In the mean-convex case, this maximum always exists. We note that the quantity $k(x)$ has an obvious geometric meaning: it is the smallest curvature of the boundary of a ball contained in Ω and touching x. In other words, it is the curvature of the largest ball contained in Ω and touching x.

Applying the same arguments that we used in the proof of proposition 2.1, it is clear that $k(x)$ is bounded from below by the greatest principal curvature $\lambda_1(x)$ of M at x. Additionally, we can characterise when we have equality: if the supremum in the definition of $k(x)$ is attained in the limit as $y \to x$, then $k(x) = \lambda_1(x)$.

Using this observation Andrews is able to reformulate the statement of Theorem 4.1 as follows:

$$\max_{M_0} \frac{k_0}{H_0} \leq \frac{1}{\delta} \Longrightarrow \max_{M_t} \frac{k_t}{H_t} \leq \frac{1}{\delta} \ \text{ for } \ t \in [0, T)$$

when M_t evolves by a mean-convex MCF. This implies in particular that in order to prove the above theorem, it suffices to show that the quantity

$$\max_{M_t} \frac{k_t}{H_t} \tag{11}$$

is non-increasing in t. To investigate this question Andrews makes use of the standard evolution equation

$$\partial_t H_t = \Delta H_t + |A|^2 H_t \tag{12}$$

for the mean curvature flow (see for example [18]) and derives after some analysis the estimate

$$\partial_t k_t \leq \Delta k_t + |A|^2 k_t \tag{13}$$

in the viscosity sense. Now at $t = 0$ we have

$$k_0(x) \leq \frac{H_0(x)}{\delta}$$

for all $x \in M$, and since (12) is linear we know that H_t/δ is also a solution thereof. In essence this is saying that $k_t \leq H_t/\delta$ on the boundary of the domain of a parabolic PDE, so we may apply standard comparison results to conclude that $k_t \leq H_t/\delta$ on the entire domain. Since H_t always remains positive, this implies in particular that (11) remains bounded by $1/\delta$ for all $t < T$.

4.2 The minimal surface case

Now let $F : \Sigma \to \mathbb{S}^3$ be a minimal embedded torus as before. Since minimal surfaces are stationary solutions to mean curvature flow, it makes sense to wonder whether the techniques of the previous section can say anything in this context. An obvious difficulty which immediately arises is that minimal surfaces have vanishing mean curvature, so we cannot compare the quantities H and k anymore. Brendle's remarkable insight in [5] is that, in the case of minimal tori in the 3-sphere, we can instead compare k with the greatest principal curvature, and that moreover this comparison can give us information about the curvature of the torus itself.

To see how this plays out, we must first take note of modifications to the methods of the previous section which arise from the spherical ambient geometry. Firstly, the definition of the quantity k reduces to

$$k(x) = \sup_{y \neq x} \frac{\langle \nu(x), F(y) \rangle}{1 - \langle F(x), F(y) \rangle}$$

because $\langle \nu, F \rangle \equiv 0$ and $\langle F, F \rangle \equiv 1$.

Secondly the analysis which yielded the viscosity estimate (13) picks up some additional terms and becomes

$$\partial_t k_t \leq \Delta k_t - \frac{|\nabla k_t|^2}{k_t} + (|A|^2 - 2)k_t$$

which, since our solution is actually stationary, implies

$$0 \leq \Delta k - \frac{|\nabla k|^2}{k} + (|A|^2 - 2)k$$

again in the viscosity sense.

Now let $\Psi(x)$ be the maximum principal curvature of Σ at x - by the assumption of minimality, we have $2\Psi^2 = |A|^2$. In particular Ψ is strictly positive by Theorem 3.2 and satisfies the Simons-type identity of Proposition 2.3, so it seems we may deploy a comparison technique to k/Ψ analogous to that in §4.1. The usefulness of analysing this ratio comes in part from a result of Simons in [17] which says in particular that any minimal torus in $\mathbb{S}^3$ has either

$\Psi \equiv 1$ or $\Psi(x) > 1$ for some $x \in \Sigma$. In investigating it from this angle, one is therefore drawn to studying the quantity

$$\kappa = \sup_{\Sigma} \frac{|k|}{\Psi}$$

and the function

$$Z(x,y) = \kappa\Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle$$

which is of course the function Brendle studies in both the $\kappa = 1$ and the $\kappa > 1$ cases. We are also in a position

In this way, we can at the very least get a glimpse of the origins of the main players in [5].

5 Further directions

The proof by Brendle of the Lawson Conjecture that we have examined here was somehow innovative in the techniques involved. Thus, it opened the door to try and replicate those in order to prove similar conjectures in the field, or even to work on another not so related problems. In this section, we will take a look at some of the results that appeared shortly after this to evaluate the importance and repercussion of the paper that we have studied, which in turn, justifies our interest in understanding it very deeply.

The first application of the techniques showed in Brendle's proof that we would like to remark, is the paper [2] by Andrews and Li. There, they proved that any **constant mean curvature (CMC) embedded torus** in the three-dimensional sphere is axially symmetric. This allowed to confirm the Pinkall-Sterling conjecture, which was introduced in 1970 (see [20]) and states that all CMC embedded tori in the 3-sphere are surfaces of revolution. Observe that minimal surfaces in the sphere, such as the Clifford torus, are particular cases of that situation, having in fact constantly vanishing mean curvature.

Additionally, thanks to this axial symmetry and using ODE methods, in that same article they were able to classify all the CMC tori embedded in the sphere in terms of the constant value of its mean curvature. Concerning this line of research, some other advances have been made with the intention of going to more general settings. First, Brendle himself [7] was able to extend the results to *Alexandrov embedded surfaces*, which are immersed surface lying on the boundary of an immersed 3-manifold. Some time after, he also applied these ideas to *Weingarten surfaces* [8] in which a suitable function of principal curvatures is constant. In this context, Brendle was able to prove the same rotational symmetry result and in some cases that they are in fact Clifford tori.

Now let us turn to the case of **higher dimensional spheres**. It is quite remarkable that we know so much about the case of the 3-dimensional sphere, while so little is known about the case of higher dimensional ones. In [14], the paper in which Lawson showed that there are no umbilic points in a embedded torus in $\mathbb{S}^3$, he also proved that closed surfaces of any orientable type can be minimally embedded in this 3-dimensional sphere, and that for non-orientable ones, there always exists a minimal immersion with the only exception of $\mathbb{RP}^2$. In the orientable case, he also found that these minimal embeddings are not unique for surfaces with a genus g that is not prime. Moreover, in this text we have shown that we do have uniqueness of those embeddings in the case $g = 0$ (cf. theorem 3.1) and in the case $g = 1$ (that was the objective of this project in the first place). As we said, if we turn to higher dimensional spheres, we will find that we lack any kind of characterisation of even the topological types of m-dimension manifolds that can be minimally embedded or immersed in $\mathbb{S}^{m+1}$.

An interesting case, and the natural successor of the one we have studied in this work, would be the **four dimensional sphere**. Until about a decade ago only three topological types were known to be able to be minimally embedded or immersed in $\mathbb{S}^4$:

1. the 3-dimensional sphere, which admits a totally geodesic realisation,
2. the product $\mathbb{S}^2 \times \mathbb{S}^1$,
3. and a quotient of $SO(3)$ by a Cartan isoparametric hypersurface.

In fact, the first two are actually know to be minimally embeddable in $\mathbb{S}^4$ in infinitely many, pairwise non-isometric, different ways (see [11], [12]).

To the best of our knowledge, the first advances regarding this question, with respect to what we just stated, were given in [9], a paper released last year by Alessandro Carlotto and Mario B. Schulz. In that work, the authors are concerned with higher dimensional tori, or rather *hypertori*, that is, products of

$$T^m = \mathbb{S}^1 \times ... \times \mathbb{S}^1$$

$m \geq 2$ spheres $\mathbb{S}^1$. In particular, they prove that there exists a minimal embedding of T^3 into the four dimensional sphere $\mathbb{S}^4$, as well as infinitely many pairwise non-isometric immersed ones. This result is certainly similar to the situation in the 3-dimensional sphere. However, the techniques used in [9] to prove this fact are very different from the ones that Brendle used in the paper we have reviewed here. There, they employ equivariant techniques, about which we will not say much, since they could perfectly be another mini-project on their own.

Actually, the fact that we just stated is just a corollary (case $n = 2$) of the more general results that are proven in [9], which are the following.

Theorem 5.1. *For any $2 \leq n \in \mathbb{N}$ there exists a minimal embedding of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1$ in the round sphere $\mathbb{S}^{2n}$.*

Theorem 5.2. *Let $n \in \{2, 3\}$. Then, there exist infinitely many, pairwise non-isometric, minimal immersions of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^1$ in $\mathbb{S}^{2n}$.*

As a final remark, this paper that we are mentioning just now is also concerned with **Chern's spherical Bernstein conjecture**, that is, to answer whether a minimal embedding of the

m-dimensional sphere into $\mathbb{S}^{m+1}$ must be totally geodesic, and thus, a standard equator. Of course, we know by theorem 3.1, that we have an affirmative answer for $m = 2$, that is, for the embedding of $\mathbb{S}^2$ into the 3-dimensional sphere $\mathbb{S}^3$. Hsiang was able to disprove in [11] such conjecture in the cases of embeddings of $\mathbb{S}^3$ and $\mathbb{S}^5$ into the spheres of dimension 4 and 6, respectively. In [9], for the proof of theorem 5.2 that we just stated, the authors need to develop a huge amount of work, which ends up being useful to give a simpler proof of Hsiang's result. In particular, they found the following.

Theorem 5.3. *Let $n \in \{2, 3\}$. Then, there exist infinitely many, pairwise non-isometric, minimal non-equatorial embeddings of $\mathbb{S}^{2n-1}$ in $\mathbb{S}^{2n}$.*

Appendices

A The Clifford torus is minimal

We should probably check that the Clifford torus is actually minimal. This is a basic result that only involves some computations. In particular, we will check that its principal curvatures are constantly 1 and -1, so that the trace of the second fundamental form vanishes everywhere.

Taking into account equation 1, which describes the Clifford torus seen in $\mathbb{T} \subset \mathbb{S}^3 \subset \mathbb{R}^4$, we can easily find the following local chart for the torus

$$\varphi : (u, v) \mapsto \frac{1}{\sqrt{2}} (\cos(u), \sin(u), \cos(v), \sin(v))$$

defined in a different open set $U \subset \mathbb{R}^2$ depending on the point $p \in \mathbb{T}$ that we are considering. With that in mind, we have that a basis of the tangent space to the torus is given by the vectors

$$\partial_u = \frac{1}{\sqrt{2}}(-\sin(u), \cos(u), 0, 0), \qquad \partial_v = \frac{1}{\sqrt{2}}(0, 0 - \sin(u), \cos(v)).$$

We are interested in finding a unit normal of the Clifford torus, that is a vector at each point $p \in \mathbb{T}$ lying in $T_p\mathbb{S}^3$ which is perpendicular to the previous vectors ∂_u and ∂_v. Since the tangent space $T_p\mathbb{S}^3$ is precisely

$$\ker(d\varphi(u, v))$$

we easily find that such a unit normal is given by

$$\nu(u, v) = \frac{1}{\sqrt{2}}(\cos(u), \sin(u), -\cos(v), -\sin(v))$$

for each $(u, v) \in U$. Thus, its differential is

$$d\nu(u, v) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & \sin(v) \\ 0 & -\cos(v) \end{pmatrix}.$$

Lastly, using that

$$\mathrm{II}(w_1, w_2) = -\langle d\nu(w_1), w_2 \rangle \nu = h(w_1, w_2)\nu$$

we can conclude what we wanted: the matrix of the second fundamental form is diagonal at every point and the principal curvatures are

$$h(\partial_u, \partial_u) = -1, \qquad h(\partial_v, \partial_v) = 1$$

so that the trace of the second fundamental form vanishes everywhere. Thus, $\mathbb{T}$ is minimal.

B Some elementary facts

Brendle's argument makes repeated use of a number of basic relations which are not explicitly stated. In order to make up for the authors' relatively low fluency with these, we have found it useful to record them outright.

Lemma B.1. *Let $\bar{x} \in \Sigma$ and choose geodesic coordinates (x_1, x_2) near x. Then*

$$\frac{\partial \nu}{\partial x_i}(\bar{x}) = -h_i^k(\bar{x})\frac{\partial F}{\partial x_k}(\bar{x}) \tag{14}$$

where the h_i^k are the components of h in the local frame (∂_1, ∂_2).

Proof. Notice that $\partial_i \nu$ is tangent to Σ, since if γ is a geodesic in Σ corresponding to the coordinate direction x_i then

$$0 = \frac{d}{dt}\langle \nu(\gamma(t)), \nu(\gamma(t))\rangle = 2\langle \nabla^0_{\gamma'}\nu(\gamma(t)), \nu(\gamma(t)) = 2\left\langle \frac{\partial \nu}{\partial x_i}, \nu \right\rangle.$$

On the other hand, because $F \perp \nu$ and $\partial_i F \perp \nu$ everywhere we have

$$0 = \frac{d}{dt}\langle \nu(\gamma(t)), F(\gamma(t))\rangle = \left\langle \frac{\partial \nu}{\partial x_i}, F \right\rangle + \left\langle \nu, \frac{\partial F}{\partial x_i} \right\rangle = \left\langle \frac{\partial \nu}{\partial x_i}, F \right\rangle.$$

Hence the projection of $\partial_i \nu$ onto the normal bundle of Σ thought of as a subspace of $\mathbb{R}^4$ is zero, which implies exactly that $\partial_i \nu \in T\Sigma$ (the fact that we are using geodesic coordinates is not necessary for this part). Then we have

$$h_i^k = h(\partial_i, \partial_k) = -\langle \nabla^0_{\partial_i}\nu, F_* \partial_k\rangle = -\left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_k} \right\rangle$$

which is saying precisely that $-h_i^k(\bar{x})$ is the $\partial_k F(\bar{x})$-component of $\partial_i \nu(\bar{x})$ (clearly the vectors $\partial_k F(\bar{x})$ form a basis of $T_{\bar{x}}\Sigma$). □

Lemma B.2. *With $\bar{x}$ and (x_1, x_2) as above we have*

$$\frac{\partial}{\partial x_1}h_1^k(\bar{x}) + \frac{\partial}{\partial x_2}h_2^k(\bar{x}) = 0 \tag{15}$$

for $k = 1, 2$.

Proof. The Codazzi equations in this context read

$$(\nabla^\Sigma_X h)(Y, Z) - (\nabla^\Sigma_Y h)(X, Z) = 0$$

for all vector fields X, Y, Z on Σ. In particular $(\nabla^\Sigma_{\partial_i} h)(\partial_j, \partial_k) = (\nabla^\Sigma_{\partial_j} h)(\partial_i, \partial_k)$ for $i, j, k = 1, 2$. Since these are geodesic coordinates centered at $\bar{x}$, we get

$$\frac{\partial h_j^k}{\partial x_i}(\bar{x}) = \frac{\partial h_i^k}{\partial x_j}(\bar{x}).$$

By the minimality of the embedding $h_1^1 + h_2^2 = 0$ everywhere, and the symmetry of the second fundamental form implies that $h_1^2 = h_2^1$ too, so we conclude that

$$\frac{\partial}{\partial x_1} h_1^k(\bar{x}) + \frac{\partial}{\partial x_2} h_2^k(\bar{x}) = 0$$

for any k. $\square$

Lemma B.3. *Let Δ_Σ be the Laplace-Beltrami operator on Σ. Then*

$$\Delta_\Sigma F = -2F \tag{16}$$

and

$$\Delta_\Sigma \nu = -|A|^2 \nu. \tag{17}$$

Proof. Let $\bar{x} \in \Sigma$ and as usual choose orthonormal geodesic coordinates (x_1, x_2) centered at $\bar{x}$ so that the second fundamental form is diagonal in the frame (∂_1, ∂_2). Let $\tilde{h}$ denote the second fundamental form of $\mathbb{S}^3$ in $\mathbb{R}^4$ locally defined with respect to F, which we think of as a unit normal. We have

$$\begin{aligned}\frac{\partial^2 F}{\partial x_i \partial x_j} = \nabla^0_{F_*\partial_i} F_* \partial_j = \nabla^S_{F_*\partial_i} F_*\partial_j + \tilde{h}(F_*\partial_i, F_*\partial_j)F \\ = F_* \nabla^\Sigma_{\partial_i} \partial_j + h(\partial_i, \partial_j)\nu + \tilde{h}(F_*\partial_i, F_*\partial_j)F\end{aligned}$$

and hence

$$\Delta_\Sigma F(\bar{x}) = 2H(\bar{x})\nu + \sum_{i=1}^{2} \tilde{h}(F_*\partial_i, F_*\partial_i)F(\bar{x}).$$

because the first term vanishes by the normality of our coordinates. Minimality means that $H \equiv 0$, and

$$\tilde{h}(F_*\partial_i, F_*\partial_i) = -\langle \nabla^0_{F_*\partial_i} F, F_*\partial_i \rangle = -\left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i} \right\rangle = -1$$

by orthonormality, so we conclude that $\Delta_\Sigma F = -2F$. For (17), we differentiate (14) to get

$$\frac{\partial^2 \nu}{\partial x_i^2} = -\frac{\partial}{\partial x_i}(h_i^k) \cdot \frac{\partial F}{\partial x_k} - h_i^k \cdot \frac{\partial^2 F}{\partial x_i \partial x_k}.$$

The first term vanishes by (15) and we can use our computation above to turn what remains into

$$\begin{aligned}\Delta_\Sigma \nu(\bar{x}) &= -\sum_{i=1}^{2} h_i^k(\bar{x}) \left(h_i^k(\bar{x})\nu + \tilde{h}(F_*\partial_i, F_*\partial_k)F(\bar{x}) \right) \\ &= -|A|^2 \nu(\bar{x}) + H(\bar{x})F = -|A|^2 \nu(\bar{x})\end{aligned}$$

as required, where we used orthonormality and the fact that h is diagonal at $\bar{x}$ $\square$

Lemma B.4. *Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a unit speed geodesic. Then*

$$\gamma''(t) = -\gamma(t) - h(\gamma'(t), \gamma'(t))\nu(\gamma(t)). \tag{18}$$

Proof. Writing the acceleration in terms of ∇^0 and decomposing into tangent and normal directions to $\mathbb{S}^3$ at $\gamma(t)$, we have

$$\gamma''(t) = \nabla^0_{\gamma'}\gamma'(t) = \nabla^S_{\gamma'}\gamma'(t) + \langle \nabla^0_{\gamma'}\gamma'(t), \gamma(t)\rangle\gamma(t)$$

where we have used the fact that $N_{\gamma(t)}\mathbb{S}^3$ is spanned by $\gamma(t)$. We can simplify the second term by calculating

$$0 = \frac{d}{dt}||\gamma(t)||^2 = 2\langle \gamma'(t), \gamma(t)\rangle$$

from which it follows that

$$0 = \frac{d}{dt}\langle \gamma'(t), \gamma(t)\rangle = \langle \nabla^0_{\gamma'}\gamma'(t), \gamma(t)\rangle + ||\gamma'(t)||^2 = \langle \nabla^0_{\gamma'}\gamma'(t), \gamma(t)\rangle + 1.$$

Hence

$$\gamma''(t) = \nabla^S_{\gamma'}\gamma'(t) - \gamma(t).$$

Now we can further decompose $\nabla^S_{\gamma'}\gamma'(t)$ with respect to Σ, which yields

$$\begin{aligned}\nabla^S_{\gamma'}\gamma'(t) &= \nabla^\Sigma_{\gamma'}\gamma'(t) - h(\gamma'(t), \gamma'(t))\nu(\gamma(t)) \\ &= -h(\gamma'(t), \gamma'(t))\nu(\gamma(t))\end{aligned}$$

because γ is a geodesic in Σ. $\square$

Proposition B.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be smooth. Let $a_{ij} : \Omega \to \mathbb{R}$ be smooth functions so that $(a_{ij}(x))$ is a positive-definite matrix for each $x \in \Omega$. Then there exist vector fields $X_1, \ldots, X_n$ such that*

$$\sum_{i,j} a_{ij}(x) D^2u(\partial_i, \partial_j) = \sum_k D^2u(X_k, X_k)$$

Proof. Since $A(x) := (a_{ij}(x))$ is always positive definite, at each $x \in \Omega$ we can consider the inner product associated to $A(x)$. Applying the Gram-Schmidt process for this inner product at every point to the basis $\partial_i(x)$ yields a collection $V_1, \ldots, V_n$ of smooth vector fields on Ω satisfying

$$V_i^T(x)A(x)V_j(x) = \delta_{ij}$$

for all $x \in \Omega$. We then define $X_k = AV_k$, and we check

$$
\begin{aligned}
D^2u(X_k, X_k) &= (\nabla_{X_k} du)(X_k) \\
&= X_k(du(X_k)) - du(\nabla_{X_k} X_k) \\
&= X_k(X_k^i \frac{\partial u}{\partial x_i}) - du(X_k^j \frac{\partial X_k^i}{\partial x_j} \partial_i) \\
&= X_k^i X_k^j \frac{\partial^2 u}{\partial x_i \partial x_j} + X_k^j \frac{\partial X_k^i}{\partial x_j} \frac{\partial u}{\partial x_i} - X_k^j \frac{\partial X_k^i}{\partial x_j} \frac{\partial u}{\partial x_i}
\end{aligned}
$$

so

$$\sum_k D^2u(X_k, X_k) = \sum_{i,j} \left(\sum_k X_k^i X_k^j \right) D^2u(\partial_i, \partial_j).$$

It therefore remains to show that

$$\sum_k X_k X_k^T = A$$

but this follows quickly after noting that

$$\sum_k X_k X_k^T V_i = \sum_k AV_k V_k^T AV_i = \sum_k AV_k \delta_{ik} = AV_i.$$

$\square$

C Computations of the second derivatives

Throughout this section, we consider an arbitrary pair of points $x \neq y$ and let (x_1, x_2) and (y_1, y_2) be geodesic normal coordinates around x and y respectively. We choose these in such a way that the second fundamental form is diagonal. We recall that

$$Z(x, y) = \kappa\Psi(x)(1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle$$

so

$$
\begin{aligned}
\frac{\partial Z}{\partial x_i}(x, y) &= \kappa \frac{\partial \Psi}{\partial x_i}(x)(1 - \langle F(x), F(y) \rangle) - \kappa\Psi(x) \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle + \left\langle \frac{\partial \nu}{\partial x_i}(x), F(y) \right\rangle \\
&= \kappa \frac{\partial \Psi}{\partial x_i}(x)(1 - \langle F(x), F(y) \rangle) - (\kappa\Psi(x) - h_i^i(x)) \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle
\end{aligned}
$$

whence it follows by minimality that

$$\sum_{i=1}^2 \frac{\partial Z}{\partial x_i}(x, y) = \kappa \sum_{i=1}^2 \frac{\partial \Psi}{\partial x_i}(x)(1 - \langle F(x), F(y) \rangle) - \kappa\Psi(x) \sum_{i=1}^2 \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle.$$

With respect to the second set of variables we get

$$\frac{\partial Z}{\partial y_i}(x,y) = -\kappa\Psi(x)\left\langle F(x), \frac{\partial F}{\partial y_i}(y)\right\rangle + \left\langle \nu(x), \frac{\partial F}{\partial y_i}(y)\right\rangle.$$

Now

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i^2}(x,y) = \kappa\frac{\partial^2\Psi}{\partial x_i^2}(x)(1-\langle F(x),F(y)\rangle) - 2\kappa\frac{\partial\Psi}{\partial x_i 9a0}(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle \\ -\kappa\Psi(x)\left\langle\frac{\partial^2 F}{\partial x_i^2}(x),F(y)\right\rangle + \left\langle\frac{\partial^2\nu}{\partial x_i^2}(x),F(y)\right\rangle\end{aligned}$$

and so

$$\begin{aligned}\sum_{i=1}^2\frac{\partial^2 Z}{\partial x_i^2}(x,y) &= \kappa\Delta_\Sigma\Psi(x)(1-\langle F(x),F(y)\rangle) - 2\kappa\sum_{i=1}^2\frac{\partial\Psi}{\partial x_i}(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle \\ &\quad -\kappa\Psi(x)\langle\Delta_\Sigma F(x),F(y)\rangle + \langle\Delta_\Sigma\nu(x),F(y)\rangle \\ &= \kappa\Delta_\Sigma\Psi(x)(1-\langle F(x),F(y)\rangle) - 2\kappa\sum_{i=1}^2\frac{\partial\Psi}{\partial x_i}(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle \\ &\quad + 2\kappa\Psi(x)\langle F(x),F(y)\rangle - |A(x)|^2\langle\nu(x),F(y)\rangle\end{aligned}$$

We also have

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i\partial y_i}(x,y) &= -\kappa\frac{\partial\Psi}{\partial x_i}(x)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle - \kappa\Psi(x)\left\langle\frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle + \left\langle\frac{\partial\nu}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\ &= -\kappa\frac{\partial\Psi}{\partial x_i}(x)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle - (\kappa\Psi(x)-h_i^i(x))\left\langle\frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle\end{aligned}$$

from which it follows by minimality that

$$\begin{aligned}\sum_{i=1}^2\frac{\partial^2 Z}{\partial x_i\partial y_i}(x,y) &= -\kappa\sum_{i=1}^2\frac{\partial\Psi}{\partial x_i}(x)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle - \kappa\Psi(x)\sum_{i=1}^2\left\langle\frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\ &= -\kappa\sum_{i=1}^2\left\langle\frac{\partial\Psi}{\partial x_i}(x)F(x)+\Psi(x)\frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle.\end{aligned}$$

Finally we compute

$$\frac{\partial^2 Z}{\partial y_i^2}(x,y) = -\kappa\Psi(x)\left\langle F(x),\frac{\partial^2 F}{\partial y_i^2}(y)\right\rangle + \left\langle\nu(x),\frac{\partial^2 F}{\partial y_i^2}(y)\right\rangle$$

so

$$\begin{aligned}\sum_{i=1}^2\frac{\partial^2 Z}{\partial y_i^2}(x,y) &= -\kappa\Psi(x)\langle F(x),\Delta_\Sigma F(y)\rangle + \langle\nu(x),\Delta_\Sigma F(y)\rangle \\ &= 2\kappa\Psi(x)\langle F(x),F(y)\rangle - 2\langle\nu(x),F(y)\rangle \\ &= -2Z(x,y) + 2\kappa\Psi(x)\end{aligned}$$

Adapting the argument of the proof of Lemma 3 of [5], we prove the following result. Let w_i be the reflection of the vector $\frac{\partial F}{\partial x_i}(x)$ across the hyperplane orthogonal to $F(x) - F(y)$, that is

$$w_i = \frac{\partial F}{\partial x_i}(x) - 2\left\langle \frac{\partial F}{\partial x_i}(x), \frac{F(x) - F(y)}{|F(x) - F(y)|} \right\rangle \frac{F(x) - F(y)}{|F(x) - F(y)|}.$$

Proposition C.1. *In the previous conditions we have*

$$\sum_{i=1}^{2} \left| w_i - \frac{\partial F}{\partial y_i}(y) \right| \leq \Lambda(x, y) \left(Z(x, y) + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial y_i}(x, y) \right| \right).$$

Proof. From Lemma 3 of [5] we know that

$$\langle w_i, \kappa\Psi(x)F(x) - \nu(x)\rangle = 2\left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle \frac{Z(x, y)}{|F(x) - F(y)|^2}$$

and

$$\left\langle \frac{\partial F}{\partial y_i}(y), \kappa\Psi(x)F(x) - \nu(x) \right\rangle = -\frac{\partial Z}{\partial y_i}(x, y).$$

Now, we have that

$$\left| w_i - \frac{\partial F}{\partial y_i}(y) \right| \leq |w_i - \kappa\Psi(x)F(x) + \nu(x)| + \left| \kappa\Psi(x)F(x) - \nu(x) - \frac{\partial F}{\partial y_i}(y) \right|$$

where, on the one hand

$$|w_i - \kappa\Psi(x)F(x) + \nu(x)|^2 = 2 + \kappa^2\Psi(x)^2 - 2\langle w_i, \kappa\Psi(x)F(x) - \nu(x)\rangle$$

and on the other hand

$$\left| \kappa\Psi(x)F(x) - \nu(x) - \frac{\partial F}{\partial y_i}(y) \right|^2 = \kappa^2\Psi(x)^2 + 2 - 2\left\langle \frac{\partial F}{\partial y_i}(y), \kappa\Psi(x)F(x) - \nu(x) \right\rangle.$$

With this, one has

$$\begin{aligned}
|w_i - \kappa\Psi(x)F(x) + \nu(x)|^2 - &\left| \kappa\Psi(x)F(x) - \nu(x) - \frac{\partial F}{\partial y_i}(y) \right|^2 \\
&= -2\langle w_i, \kappa\Psi(x)F(x) - \nu(x)\rangle + 2\left\langle \frac{\partial F}{\partial y_i}(y), \kappa\Psi(x)F(x) - \nu(x) \right\rangle \\
&= -4\left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle \frac{Z(x, y)}{|F(x) - F(y)|^2} - 2\frac{\partial Z}{\partial y_i}(x, y)
\end{aligned}$$

using the expressions above. If we now let

$$\varphi(x, y) = |w_i - \kappa\Psi(x)F(x) + \nu(x)| - \left| \kappa\Psi(x)F(x) - \nu(x) - \frac{\partial F}{\partial y_i}(y) \right|$$

then clearly

$$|w_i - \kappa\Psi(x)F(x) + \nu(x)| + \left|\kappa\Psi(x)F(x) - \nu(x) - \frac{\partial F}{\partial y_i}(y)\right|$$
$$= -4\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle \frac{Z(x,y)}{\varphi(x,y)|F(x)-F(y)|^2} - \frac{2}{\varphi(x,y)}\frac{\partial Z}{\partial y_i}(x,y)$$

and so if we define

$$\Lambda_i(x,y) = \max\left\{\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle \frac{-4}{\varphi(x,y)|F(x)-F(y)|^2}, \frac{-2}{\varphi(x,y)}\right\}$$

we can conclude that

$$\left|w_i - \frac{\partial F}{\partial y_i}(y)\right| \leq \Lambda_i(x,y)\left(|Z(x,y)| + \left|\frac{\partial Z}{\partial y_i}(x,y)\right|\right)$$

for each i. Then if $\Lambda = \Lambda_1 + \Lambda_2$ we immediately get the result. $\square$

Now, we adapt the proof of Proposition 6 on [5] in order to obtain the following.

Proposition C.2. *There exists a positive continuous function $\tilde{\Lambda}$ such that*

$$\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2}\frac{\partial^2 Z}{\partial y_i^2}(x,y)$$
$$\leq -\frac{\kappa^2-1}{\kappa}\frac{\Psi(x)}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle^2$$
$$+ \tilde{\Lambda}(x,y)\left(Z(x,y) + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial x_i}(x,y)\right| + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial y_i}(x,y)\right|\right)$$

on $\Sigma \times \Sigma \setminus \Delta$.

Proof. We examine the x_i-derivatives, recalling that

$$\frac{\partial Z}{\partial x_i}(x,y) = \kappa\frac{\partial \Psi}{\partial x_i}(x)(1-\langle F(x),F(y)\rangle) - \kappa\Psi(x)\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle + h_i^i(x)\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle$$

and

$$\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y) = \kappa\Delta_\Sigma\Psi(x)(1-\langle F(x),F(y)\rangle) - 2\kappa\sum_{i=1}^{2}\frac{\partial \Psi}{\partial x_i}(x)\left\langle \frac{\partial F}{\partial x_i}(x), F(y)\right\rangle$$
$$+ 2\kappa\Psi(x)\langle F(x),F(y)\rangle - |A(x)|^2\langle \nu(x), F(y)\rangle.$$

Note that

$$\sum_{i=1}^{2}\left(\kappa\frac{\partial\Psi}{\partial x_i}(x)(1-\langle F(x),F(y)\rangle)-\kappa\Psi(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle\right)^2$$

$$\begin{aligned}
&=\kappa^2|\nabla\Psi(x)|^2(1-\langle F(x),F(y)\rangle)^2-2\kappa^2\Psi(x)(1-\langle F(x),F(y)\rangle)\sum_{i=1}^{2}\frac{\partial\Psi}{\partial x_i}(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle\\
&\quad+\kappa^2\Psi(x)^2\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2\\
&=\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)\left(\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y)-\kappa(1-\langle F(x),F(y)\rangle)\left(\Delta_\Sigma\Psi(x)-\frac{|\nabla\Psi(x)|^2}{\Psi(x)}\right)\right)\\
&\quad-\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)(2\kappa\Psi(x)\langle F(x),F(y)\rangle-|A(x)|^2\langle\nu(x),F(y)\rangle)\\
&\quad+\kappa^2\Psi(x)^2\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2.
\end{aligned}$$

This sum is also equal to

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)-h_i^i(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle\right)^2\\
&\qquad=\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)\right)^2-2\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y)h_i^i(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle\\
&\qquad\quad+\Psi(x)^2\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2.
\end{aligned}$$

For "brevity", we let

$$G(x,y)=2\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y)h_i^i(x)\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle$$

so that

$$\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)\right)^2-G(x,y)$$

$$\begin{aligned}
&=\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)\left(\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y)-\kappa(1-\langle F(x),F(y)\rangle)\left(\Delta_\Sigma\Psi(x)-\frac{|\nabla\Psi(x)|^2}{\Psi(x)}\right)\right)\\
&\quad-\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)(2\kappa\Psi(x)\langle F(x),F(y)\rangle-|A(x)|^2\langle\nu(x),F(y)\rangle)\\
&\quad+(\kappa^2-1)\Psi(x)^2\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2.
\end{aligned}$$

Using this, we can write

$$
\begin{aligned}
\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(x,y) &= \kappa \left(\Delta_\Sigma \Psi(x) - \frac{|\nabla \Psi(x)|^2}{\Psi(x)} \right) (1 - \langle F(x), F(y) \rangle) \\
&\quad - \frac{\kappa^2 - 1}{\kappa} \frac{\Psi(x)}{1 - \langle F(x), F(y) \rangle} \sum_{i=1}^{2} \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle^2 \\
&\quad + 2\kappa \Psi(x) \langle F(x), F(y) \rangle - |A(x)|^2 \langle \nu(x), F(y) \rangle \\
&\quad + \frac{1}{\kappa \Psi(x)(1 - \langle F(x), F(y) \rangle)} \left(\sum_{i=1}^{2} \left(\frac{\partial Z}{\partial x_i}(x,y) \right)^2 - G(x,y) \right) \\
&= \kappa \left(\Delta_\Sigma \Psi(x) - \frac{|\nabla \Psi(x)|^2}{\Psi(x)} + (|A(x)|^2 - 2)\Psi(x) \right) (1 - \langle F(x), F(y) \rangle) \\
&\quad - \frac{\kappa^2 - 1}{\kappa} \frac{\Psi(x)}{1 - \langle F(x), F(y) \rangle} \sum_{i=1}^{2} \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle^2 \\
&\quad - \kappa \Psi(x) |A(x)|^2 + \kappa \Psi(x) |A(x)|^2 \langle F(x), F(y) \rangle + 2\kappa \Psi(x) - |A(x)|^2 \langle \nu(x), F(y) \rangle \\
&\quad + \frac{1}{\kappa \Psi(x)(1 - \langle F(x), F(y) \rangle)} \left(\sum_{i=1}^{2} \left(\frac{\partial Z}{\partial x_i}(x,y) \right)^2 - G(x,y) \right) \\
&= \kappa \left(\Delta_\Sigma \Psi(x) - \frac{|\nabla \Psi(x)|^2}{\Psi(x)} + (|A(x)|^2 - 2)\Psi(x) \right) (1 - \langle F(x), F(y) \rangle) \\
&\quad - \frac{\kappa^2 - 1}{\kappa} \frac{\Psi(x)}{1 - \langle F(x), F(y) \rangle} \sum_{i=1}^{2} \left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle^2 - |A(x)|^2 Z(x,y) + 2\kappa \Psi(x) \\
&\quad + \frac{1}{\kappa \Psi(x)(1 - \langle F(x), F(y) \rangle)} \left(\sum_{i=1}^{2} \left(\frac{\partial Z}{\partial x_i}(x,y) \right)^2 - G(x,y) \right).
\end{aligned}
$$

Thus, using the fact that Ψ satisfies the PDE of Proposition 2.3, we get

$$\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y)+\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial y_i^2}(x,y)\\
&\qquad\qquad= -\frac{\kappa^2-1}{\kappa}\frac{\Psi(x)}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\left\langle \frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2-|A(x)|^2 Z(x,y)+2\kappa\Psi(x)\\
&\qquad\qquad\quad+\frac{1}{\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)}\left(\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)\right)^2-G(x,y)\right)\\
&\qquad\qquad\quad+2\kappa\Psi(x)\langle F(x),F(y)\rangle-2\langle \nu(x),F(y)\rangle\\
&\qquad\qquad= -\frac{\kappa^2-1}{\kappa}\frac{\Psi(x)}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\left\langle \frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2-(|A(x)|^2+2)Z(x,y)+4\kappa\Psi(x)\\
&\qquad\qquad\quad+\frac{1}{\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)}\left(\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)\right)^2-G(x,y)\right).
\end{aligned}$$

Recall that the mixed derivatives for the function $Z(x,y)$ satisfy the following expression

$$\frac{\partial^2 Z}{\partial x_i\partial y_i}(x,y)=-\kappa\frac{\partial \Psi}{\partial x_i}(x)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle-(\kappa\Psi(x)-h_i^i(x))\left\langle \frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle.$$

Using the expression for the derivatives with respect to x_i

$$\frac{\partial Z}{\partial x_i}(x,y)=\kappa\frac{\partial \Psi}{\partial x_i}(x)(1-\langle F(x),F(y)\rangle)-(\kappa\Psi(x)-h_i^i(x))\left\langle \frac{\partial F}{\partial x_i}(x),F(y)\right\rangle$$

we get

$$\begin{aligned}
\frac{\partial^2 Z}{\partial x_i\partial y_i}(x,y)=&-\frac{1}{1-\langle F(x),F(y)\rangle}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle\\
&+\frac{1}{1-\langle F(x),F(y)\rangle}(h_i^i(x)-\kappa\Psi(x))\left\langle \frac{\partial F}{\partial x_i}(x),F(y)\right\rangle\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle\\
&+(h_i^i(x)-\kappa\Psi(x))\left\langle \frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle.
\end{aligned}$$

Since $1-\langle F(x),F(y)\rangle=\frac{1}{2}|F(x)-F(y)|^2$ and using the orthogonality of the tangent space to the sphere, we can rewrite

$$\begin{aligned}
2\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) &= -\frac{2}{1-\langle F(x),F(y)\rangle}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad - 4(h_i^i(x)-\kappa\Psi(x))\left\langle \frac{\partial F}{\partial x_i}(x),\frac{F(x)-F(y)}{|F(x)-F(y)|}\right\rangle\left\langle \frac{F(x)-F(y)}{|F(x)-F(y)|},\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad + 2(h_i^i(x)-\kappa\Psi(x))\left\langle \frac{\partial F}{\partial x_i}(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&= -\frac{2}{1-\langle F(x),F(y)\rangle}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad + 2(h_i^i(x)-\kappa\Psi(x))\left\langle w_i,\frac{\partial F}{\partial y_i}(y)\right\rangle .
\end{aligned}$$

Now

$$\left|w_i - \frac{\partial F}{\partial y_i}(y)\right|^2 = 2 - 2\left\langle w_i,\frac{\partial F}{\partial y_i}(y)\right\rangle$$

so

$$\begin{aligned}
2\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) &= -\frac{2}{1-\langle F(x),F(y)\rangle}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad + (h_i^i(x)-\kappa\Psi(x))\left(2-\left|w_i-\frac{\partial F}{\partial y_i}(y)\right|^2\right) \\
&= -\frac{2}{1-\langle F(x),F(y)\rangle}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad + 2h_i^i(x) - 2\kappa\Psi(x) - (h_i^i(x)-\kappa\Psi(x))\left|w_i-\frac{\partial F}{\partial y_i}(y)\right|^2
\end{aligned}$$

which gives

$$\begin{aligned}
2\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) &= -\frac{2}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y)\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle \\
&\quad - 4\kappa\Psi(x) - \sum_{i=1}^{2}(h_i^i(x)-\kappa\Psi(x))\left|w_i-\frac{\partial F}{\partial y_i}(y)\right|^2 .
\end{aligned}$$

If we let

$$H(x,y) = \max\left\{-\frac{2}{1-\langle F(x),F(y)\rangle}\left\langle F(x),\frac{\partial F}{\partial y_i}(y)\right\rangle, -h_i^i(x)-\kappa\Psi(x)\right\}$$

then

$$
\begin{aligned}
2\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) &\le H(x,y)\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y) - 4\kappa\Psi(x) + H(x,y)\sum_{i=1}^{2}\left|w_i - \frac{\partial F}{\partial y_i}(y)\right|^2 \\
&\le H(x,y)\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y) - 4\kappa\Psi(x) + H(x,y)\left(\sum_{i=1}^{2}\left|w_i - \frac{\partial F}{\partial y_i}(y)\right|\right)^2 \\
&\le H(x,y)\sum_{i=1}^{2}\frac{\partial Z}{\partial x_i}(x,y) - 4\kappa\Psi(x) \\
&\quad + H(x,y)\Lambda(x,y)^2\left(Z(x,y) + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial y_i}(x,y)\right|\right)^2 .
\end{aligned}
$$

Finally, we can combine everything to get

$$
\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2}\frac{\partial^2 Z}{\partial y_i^2}(x,y) \\
&\qquad \le -\frac{\kappa^2-1}{\kappa}\frac{\Psi(x)}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2 - (|A(x)|^2+2)Z(x,y) \\
&\qquad\quad + \frac{1}{\kappa\Psi(x)(1-\langle F(x),F(y)\rangle)}\left(\sum_{i=1}^{2}\left(\frac{\partial Z}{\partial x_i}(x,y)\right)^2 - G(x,y)\right) \\
&\qquad\quad + H(x,y)\sum_{i=1}^{2}\left|\frac{\partial Z}{\partial x_i}(x,y)\right| + H(x,y)\Lambda(x,y)^2\left(Z(x,y) + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial y_i}(x,y)\right|\right)^2
\end{aligned}
$$

Now recalling the definition of $G(x,y)$, one sees that every term apart from the first one may be expressed as (or possibly bounded above by) the product of a continuous function on $\Sigma\times\Sigma\setminus\Delta$ with one of

$$
Z(x,y)\,,\quad \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial x_i}(x,y)\right|\,,\quad \text{or}\quad \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial y_i}(x,y)\right|
$$

and thus there exists a positive continuous function $\tilde{\Lambda}$ such that

$$
\begin{aligned}
&\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i^2}(x,y) + 2\sum_{i=1}^{2}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{2}\frac{\partial^2 Z}{\partial y_i^2}(x,y) \\
&\qquad \le -\frac{\kappa^2-1}{\kappa}\frac{\Psi(x)}{1-\langle F(x),F(y)\rangle}\sum_{i=1}^{2}\left\langle\frac{\partial F}{\partial x_i}(x),F(y)\right\rangle^2 \\
&\qquad\quad + \tilde{\Lambda}(x,y)\left(Z(x,y) + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial x_i}(x,y)\right| + \sum_{i=1}^{2}\left|\frac{\partial Z}{\partial y_i}(x,y)\right|\right)
\end{aligned}
$$

on $\Sigma\times\Sigma\setminus\Delta$. $\square$

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