

# Problem Sheet

## Persistent Homology

### Topology and Geometry in Machine Learning

#### Lecture 6 — Solutions

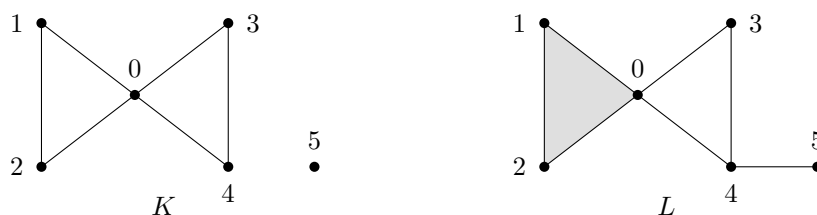
**Conventions.** All chains and homology groups have coefficients in  $\mathbb{R}$ . Orient simplices by increasing vertex order and order the simplices accordingly as we have seen in class, unless specified otherwise. Write  $\{i_0, \dots, i_k\}$  for an unoriented simplex,  $(i_0, \dots, i_k)$  for an oriented simplex and  $[z]_X$  for the homology class of a cycle in  $X$ . For vertices, abbreviate  $[(v)]_X$  to  $[v]_X$ . Matrix columns are images of domain basis vectors.

Exercises are organized by difficulty, increasing from  $*$  to  $***$ .

**Exercise 1** (Orientations, cycles and Betti numbers) Let  $K$  consist of two hollow triangles sharing vertex 0, together with an isolated vertex 5. Its edges are

$$\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{3, 4\}.$$

There are no higher-dimensional simplices. Let  $L$  be obtained by adding the filled triangle  $\{0, 1, 2\}$  and the edge  $\{4, 5\}$ .



- (a,  $*$ ) Express  $(2, 1, 0)$ ,  $(1, 2, 0)$  and  $(2, 0, 1)$  as signed multiples of  $(0, 1, 2)$  (by changing the orientation of the latter simplex). Compute  $\partial_2(0, 1, 2)$  and  $\partial_2(2, 1, 0)$ . Check directly that  $\partial_1 \partial_2(0, 1, 2) = 0$ .
- (b,  $**$ ) Write the matrices  $B_1^K$ ,  $B_1^L$  and  $B_2^L$  of the boundary operators. Compute their ranks (equivalently, the dimension of their images or the number of linearly independent columns) and find bases of  $\ker B_1^K$ ,  $\ker B_1^L$ ,  $\text{im } B_2^L$  and  $\text{im } B_2^K$ . Use

$$a = (0, 1) + (1, 2) - (0, 2), \quad b = (0, 3) + (3, 4) - (0, 4).$$

- (c,  $*$ ) Determine  $H_k(K)$  and  $H_k(L)$ , with explicit generators, and all their Betti numbers.
- (d,  $**$ ) Reverse only the orientation of edge  $(0, 2)$ . Describe how the corresponding row of  $B_2^L$ , column of  $B_1^L$  and coordinate of  $a$  change. Why do the Betti numbers stay unchanged?

#### Solution.

- (a) The signs are  $(2, 1, 0) = -(0, 1, 2)$ ,  $(1, 2, 0) = (0, 1, 2)$ ,  $(2, 0, 1) = (0, 1, 2)$ . Also

$$\partial_2(0, 1, 2) = (1, 2) - (0, 2) + (0, 1) = a, \quad \partial_2(2, 1, 0) = -a.$$

Applying  $\partial_1$  gives  $((2) - (1)) - ((2) - (0)) + ((1) - (0)) = 0$ .

(b) In the vertex basis  $\{(0), (1), (2), (3), (4), (5)\}$  and edge basis

$$\{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (3, 4)\},$$

$$B_1^K = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1^L = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus  $B_1^L$  appends the boundary column for  $(4, 5)$  to  $B_1^K$ . Its edge basis is

$$\{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (3, 4), (4, 5)\}.$$

$$B_2^L = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The ranks are

$$\text{rank } B_1^K = 4, \quad \text{rank } B_1^L = 5, \quad \text{rank } B_2^L = 1.$$

For both complexes,  $\ker B_1^K = \ker B_1^L = \text{span}(a, b)$  since

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = 0$$

reduces to

$$\begin{cases} u + v + w + x = 0 \\ u - y = 0 \\ v + y = 0 \\ w - z = 0 \end{cases}$$

which has space of solutions spanned by:

$$\{(1, -1, 0, 0, 1, 0), (0, 0, 1, -1, 0, 1)\}$$

which, in the basis of edges of  $L$ , are precisely  $a = (0, 1) - (0, 2) + (1, 2)$  and  $b = (0, 3) - (0, 4) + (3, 4)$ .

The computations for  $B_1^L$  proceed similarly, but adding a last equation  $t = 0$ , with  $t$  the last coordinate of a vector in the chain space accompanying  $(4, 5)$ . Hence, we append a zero coordinate for  $(4, 5)$  in the spanning vectors above, which still leads to  $a$  and  $b$ .

There are no 2-simplices in  $K$ , so  $\text{im } B_2^K = 0$ , whereas

$$\text{im } B_2^L = \text{span}(\partial_2^K(0, 1, 2)) = \text{span}(a).$$

(c) The previous computations lead to the following homology groups and Betti numbers:

| $X$ | $H_0(X)$                | $H_1(X)$                | $H_k(X), k \geq 2$ | $(\beta_0, \beta_1, \beta_2, \dots)$ |
|-----|-------------------------|-------------------------|--------------------|--------------------------------------|
| $K$ | $\text{span}([0], [5])$ | $\text{span}([a], [b])$ | 0                  | $(2, 2, 0, \dots)$                   |
| $L$ | $\text{span}([0])$      | $\text{span}([b])$      | 0                  | $(1, 1, 0, \dots)$                   |

(d) The new basis vector is  $(2, 0) = -(0, 2)$ , so the column of  $B_1^L$  and row of  $B_2^L$  indexed by this edge both need to change sign. That way, the coordinate of  $a$  changes from  $-1$  to  $+1$ . These are invertible basis changes. The notion of dimension is independent of the choice of basis, it has to do with the number of vectors in a basis, not with the vectors themselves. Thus, the ranks and Betti numbers remain unchanged.

**Exercise 2** (Matrices of induced maps: inclusion and identification) Use  $K, L, a, b$  from Exercise 1. For a simplicial map  $f : K \rightarrow L$ , recall that the induced map  $f_{\#,k} : C_k(K) \rightarrow C_k(L)$  is defined the basis of oriented simplices as

$$f_{\#,k}(v_0, \dots, v_k) = \begin{cases} (f(v_0), \dots, f(v_k)), & \text{if the images of the vertices are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

Reorder distinct image vertices into the target orientation, keeping track of the sign if it changes. Denote  $f_{*,k} : H_k(K) \rightarrow H_k(L)$  the induced map on homology.

**Part I: an inclusion.** Let  $i : K \hookrightarrow L$  be the inclusion.

- (a, \*) Write the matrices  $I_0, I_1, I_2$  of  $i_{\#,k}$  for  $k = 0, 1, 2$ , including the dimensions of any matrices with zero columns. Verify that they commute with the boundary map matrices, i.e. that

$$B_1^L I_1 = I_0 B_1^K, \quad B_2^L I_2 = I_1 B_2^K.$$

- (b, \*\*) Compute the matrices of the induced maps  $i_{*,0}$  and  $i_{*,1}$  in the following ordered homology bases:

|       |                    |             |
|-------|--------------------|-------------|
|       | $K$                | $L$         |
| $H_0$ | $\{[0]_K, [5]_K\}$ | $\{[0]_L\}$ |
| $H_1$ | $\{[a]_K, [b]_K\}$ | $\{[b]_L\}$ |

Describe  $i_{*,2}$ . Why can an inclusion be injective on chains but fail to be injective on homology? Identify a nonzero kernel element in each of degrees 0 and 1.

**Part II: a map that is not an inclusion.** Let  $Y$  be the hollow triangle with ordered vertices  $u < v < w$  and edges  $\{u, v\}, \{u, w\}, \{v, w\}$ . Define  $f : K \rightarrow Y$  by

|        |     |     |     |     |     |     |
|--------|-----|-----|-----|-----|-----|-----|
| $x$    | 0   | 1   | 2   | 3   | 4   | 5   |
| $f(x)$ | $u$ | $w$ | $v$ | $u$ | $v$ | $w$ |

Set  $c = (u, v) + (v, w) - (u, w)$ .

- (c, \*) Check that  $f$  is simplicial. Which edges collapse to vertices? Which edges acquire a minus sign when their images are written in the target basis?
- (d, \*\*) Compute the matrices  $F_0$  and  $F_1$  of  $f_{\#,0}$  and  $f_{\#,1}$ , and verify  $B_1^Y F_1 = F_0 B_1^K$ .
- (e, \*\*) Compute the matrices of  $f_{*,0}$  and  $f_{*,1}$  using the above bases for  $K$  and the bases  $\{[u]_Y\}$  and  $\{[c]_Y\}$  for  $Y$ . Determine their kernels and images.
- (f, \*\*) Can the same vertex map extend to a simplicial map  $L \rightarrow Y$ ?

**Solution.**

- (a) The chain matrices are

$$I_0 = \text{Id}_6, \quad I_1 = \begin{pmatrix} \text{Id}_6 \\ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{pmatrix}, \quad I_2 = 0.$$

For the commutativity property: the first identity selects the first six columns of  $B_1^L$ , which are  $B_1^K$ . The second has both sides equal to 0.

- (b) Both generators of  $H_0(K)$  land in the single generator of  $H_0(L)$ , therefore

$$i_{*,0} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

On the other hand,  $a$  bounds  $(0, 1, 2)$  in  $L$ , so its class becomes 0 in  $H_1(L)$ ; while  $b$  remains a cycle that is not a boundary of any triangle in  $L$ , hence:

$$i_{*,1} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The map  $i_{*,2}$  is  $0 \rightarrow 0$ .

Kernels are spanned by  $[5]_K - [0]_K$  and  $[a]_K$ , for  $i_{*,0}$  and  $i_{*,1}$ , respectively.

An injective chain map can send a nonboundary cycle to a boundary in the larger complex, thus creating an element in the kernel of the induced map at the level of homology, which ceases to be injective.

(c) Every edge maps to an edge of  $Y$  or a single vertex, so the map is simplicial. Edge  $\{0, 3\}$  collapses. The oriented edge  $(1, 2)$  maps to  $(w, v) = -(v, w)$ ; all other noncollapsed edge images have positive signs.

(d) In the stated bases,

$$F_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_1^Y = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Multiplication on either side of the chain-map identity gives

$$B_1^Y F_1 = F_0 B_1^K = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

(e) We have  $f_{\#,1}(a) = (u, w) - (v, w) - (u, v) = -c$  and  $f_{\#,1}(b) = 0 + (u, v) - (u, v) = 0$ . Also  $[f(0)] = [u] = [w] = [f(5)]$  in  $H_0(Y)$ . Hence

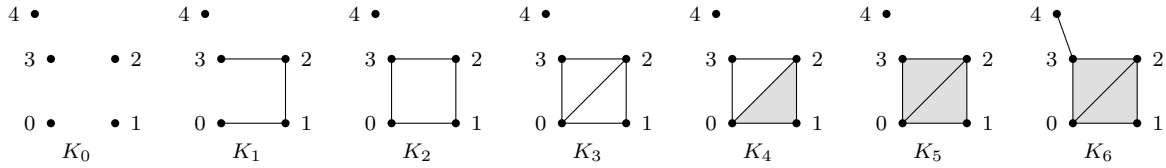
$$f_{*,0} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad f_{*,1} = \begin{pmatrix} -1 & 0 \end{pmatrix}.$$

The kernels are  $\text{span}([5] - [0])$  and  $\text{span}([b])$ , respectively, and both maps are surjective.

(f) No:  $\{0, 1, 2\}$  would map onto  $\{u, v, w\}$ , which is not a simplex of  $Y$ .

**Exercise 3** (A barcode by choosing compatible generators) Consider  $K_0 \subseteq \cdots \subseteq K_6$  on vertices  $0, 1, 2, 3, 4$ . Begin with five isolated vertices and add the simplices in the table, retaining everything added earlier. No other simplices are present.

| $t$ | New simplices (in addition to their faces) |
|-----|--------------------------------------------|
| 0   | $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}$        |
| 1   | $\{0, 1\}, \{1, 2\}, \{2, 3\}$             |
| 2   | $\{0, 3\}$                                 |
| 3   | $\{0, 2\}$                                 |
| 4   | $\{0, 1, 2\}$                              |
| 5   | $\{0, 2, 3\}$                              |
| 6   | $\{3, 4\}$                                 |



Use half-open barcode intervals  $[b, d)$ : a class born at  $b$  and killed upon entering  $K_d$  is alive for  $[b, d - 1]$ .

- (a, \*) Compute  $\beta_0(K_t)$ ,  $\beta_1(K_t)$ ,  $\beta_2(K_t)$  at each integer stage. Identify when components merge and when cycles appear or become boundaries.
- (b, \*\*) First use the basis of vertex homology classes for  $H_0$  and write the matrices of the consecutive maps. Then replace this basis of  $H_0(K_0)$  by

$$[0], [1] - [0], [2] - [0], [3] - [0], [4] - [0].$$

Choose compatible bases at later stages so that each generator maps to the same generator or to zero.

- (c, \*\*) Define

$$q = (0, 1) + (1, 2) + (2, 3) - (0, 3), \quad a = (0, 1) + (1, 2) - (0, 2), \quad b = (0, 2) + (2, 3) - (0, 3).$$

At which stages are these chains defined and nonzero class in homology? Show that  $q = a + b$  at stage 3. Compute the consecutive  $H_1$  maps using  $\{[q]\}$  at stage 2,  $\{[a], [b]\}$  at stage 3 and  $\{[b]\}$  at stage 4.

- (d, \*\*\*) Replace the basis at stage 3 by one containing the image of  $[q]$ . Complete it so that one generator dies at stage 4 and the other survives to stage 4.

- (e,\*) Draw the  $H_0$  and  $H_1$  barcodes.

**Solution.**

- (a) The Betti numbers are

| $t$       | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|---|---|
| $\beta_0$ | 5 | 2 | 2 | 2 | 2 | 2 | 1 |
| $\beta_1$ | 0 | 0 | 1 | 2 | 1 | 0 | 0 |
| $\beta_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

At stage 1 the first four vertices form a tree. At stage 2 its square closes; the diagonal creates a second independent cycle at stage 3. The two triangles kill one cycle each at stages 4 and 5. Vertex 4 joins the other component at stage 6.

(b) Use component bases  $\{[0], [1], [2], [3], [4]\}$  at stage 0,  $\{[0], [4]\}$  at stages 1 through 5, and  $\{[0]\}$  at stage 6. The consecutive matrices are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{Id}_2, \text{Id}_2, \text{Id}_2, \text{Id}_2, \quad \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

With the difference basis in the question, retain  $\{[0], [4] - [0]\}$  at stages 1 through 5 and  $\{[0]\}$  at stage 6. The first three differences die at stage 1, the last at stage 6, and  $[0]$  survives forever.

(c) The chain  $q$  is defined from stage 2 onward and has nonzero class at stages 2, 3, 4. Chains  $a, b$  are defined from stage 3 onward;  $[a]$  is nonzero only at stage 3, and  $[b]$  at stages 3 and 4. The relation  $q = a + b$  is a chain identity, it shows that these three vectors are linearly dependent, so we only need to keep two of them in a basis. From stage 2 to 3 the matrix is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

from stage 3 to 4 it is

$$\begin{pmatrix} 0 & 1 \end{pmatrix};$$

from stage 4 to 5 it is 0. All other  $H_1$  maps are zero.

(d) Choose  $\{[q], [a]\}$  at stage 3. These are independent since  $q = a + b$ . At stage 4 choose  $[q] = [b]$ , since  $a = \partial(0, 1, 2)$  there. The nontrivial induced map matrices become

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

from stage 2 to 3, and again

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

from stage 3 to 4.

(e) From the previous computations, the barcodes are:

