

Problem Sheet

Persistent Homology

Topology and Geometry in Machine Learning

Lecture 6

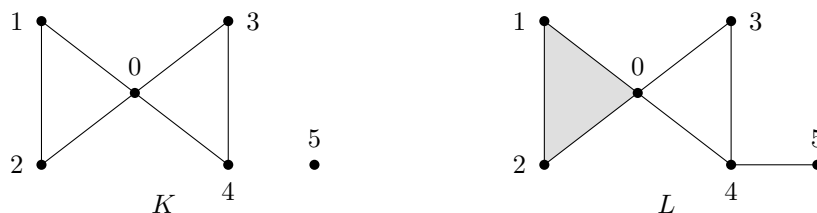
Conventions. All chains and homology groups have coefficients in $\mathbb{R}$. Orient simplices by increasing vertex order and order the simplices accordingly as we have seen in class, unless specified otherwise. Write $\{i_0, \dots, i_k\}$ for an unoriented simplex, $(i_0, \dots, i_k)$ for an oriented simplex and $[z]_X$ for the homology class of a cycle in X . For vertices, abbreviate $[(v)]_X$ to $[v]_X$. Matrix columns are images of domain basis vectors.

Exercises are organized by difficulty, increasing from $*$ to $***$.

Exercise 1 (Orientations, cycles and Betti numbers) Let K consist of two hollow triangles sharing vertex 0, together with an isolated vertex 5. Its edges are

$$\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{3, 4\}.$$

There are no higher-dimensional simplices. Let L be obtained by adding the filled triangle $\{0, 1, 2\}$ and the edge $\{4, 5\}$.



- (a, $*$) Express $(2, 1, 0)$, $(1, 2, 0)$ and $(2, 0, 1)$ as signed multiples of $(0, 1, 2)$ (by changing the orientation of the latter simplex). Compute $\partial_2(0, 1, 2)$ and $\partial_2(2, 1, 0)$. Check directly that $\partial_1\partial_2(0, 1, 2) = 0$.
- (b, $**$) Write the matrices B_1^K , B_1^L and B_2^L of the boundary operators. Compute their ranks (equivalently, the dimension of their images or the number of linearly independent columns) and find bases of $\ker B_1^K$, $\ker B_1^L$, $\text{im } B_2^L$ and $\text{im } B_2^K$. Use

$$a = (0, 1) + (1, 2) - (0, 2), \quad b = (0, 3) + (3, 4) - (0, 4).$$

- (c, $*$) Determine $H_k(K)$ and $H_k(L)$, with explicit generators, and all their Betti numbers.
- (d, $**$) Reverse only the orientation of edge $(0, 2)$. Describe how the corresponding row of B_2^L , column of B_1^L and coordinate of a change. Why do the Betti numbers stay unchanged?

Exercise 2 (Matrices of induced maps: inclusion and identification) Use K, L, a, b from Exercise 1. For a simplicial map $f : K \rightarrow L$, recall that the induced map $f_{\#,k} : C_k(K) \rightarrow C_k(L)$ is defined the basis of oriented simplices as

$$f_{\#,k}(v_0, \dots, v_k) = \begin{cases} (f(v_0), \dots, f(v_k)), & \text{if the images of the vertices are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

Reorder distinct image vertices into the target orientation, keeping track of the sign if it changes. Denote $f_{*,k} : H_k(K) \rightarrow H_k(L)$ the induced map on homology.

Part I: an inclusion. Let $i : K \hookrightarrow L$ be the inclusion.

- (a, *) Write the matrices I_0, I_1, I_2 of $i_{\#,k}$ for $k = 0, 1, 2$, including the dimensions of any matrices with zero columns. Verify that they commute with the boundary map matrices, i.e. that

$$B_1^L I_1 = I_0 B_1^K, \quad B_2^L I_2 = I_1 B_2^K.$$

- (b, **) Compute the matrices of the induced maps $i_{*,0}$ and $i_{*,1}$ in the following ordered homology bases:

	K	L
H_0	$\{[0]_K, [5]_K\}$	$\{[0]_L\}$
H_1	$\{[a]_K, [b]_K\}$	$\{[b]_L\}$.

Describe $i_{*,2}$. Why can an inclusion be injective on chains but fail to be injective on homology? Identify a nonzero kernel element in each of degrees 0 and 1.

Part II: a map that is not an inclusion. Let Y be the hollow triangle with ordered vertices $u < v < w$ and edges $\{u, v\}, \{u, w\}, \{v, w\}$. Define $f : K \rightarrow Y$ by

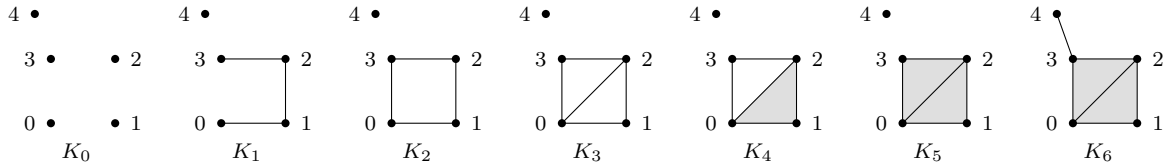
x	0	1	2	3	4	5
$f(x)$	u	w	v	u	v	w .

Set $c = (u, v) + (v, w) - (u, w)$.

- (c, *) Check that f is simplicial. Which edges collapse to vertices? Which edges acquire a minus sign when their images are written in the target basis?
- (d, **) Compute the matrices F_0 and F_1 of $f_{\#,0}$ and $f_{\#,1}$, and verify $B_1^Y F_1 = F_0 B_1^K$.
- (e, **) Compute the matrices of $f_{*,0}$ and $f_{*,1}$ using the above bases for K and the bases $\{[u]_Y\}$ and $\{[c]_Y\}$ for Y . Determine their kernels and images.
- (f, **) Can the same vertex map extend to a simplicial map $L \rightarrow Y$?

Exercise 3 (A barcode by choosing compatible generators) Consider $K_0 \subseteq \cdots \subseteq K_6$ on vertices $0, 1, 2, 3, 4$. Begin with five isolated vertices and add the simplices in the table, retaining everything added earlier. No other simplices are present.

t	New simplices (in addition to their faces)
0	$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}$
1	$\{0, 1\}, \{1, 2\}, \{2, 3\}$
2	$\{0, 3\}$
3	$\{0, 2\}$
4	$\{0, 1, 2\}$
5	$\{0, 2, 3\}$
6	$\{3, 4\}$



Use half-open barcode intervals $[b, d)$: a class born at b and killed upon entering K_d is alive for $[b, d - 1]$.

- (a, *) Compute $\beta_0(K_t)$, $\beta_1(K_t)$, $\beta_2(K_t)$ at each integer stage. Identify when components merge and when cycles appear or become boundaries.
- (b, **) First use the basis of vertex homology classes for H_0 and write the matrices of the consecutive maps. Then replace this basis of $H_0(K_0)$ by

$$[0], [1] - [0], [2] - [0], [3] - [0], [4] - [0].$$

Choose compatible bases at later stages so that each generator maps to the same generator or to zero.

- (c, **) Define

$$q = (0, 1) + (1, 2) + (2, 3) - (0, 3), \quad a = (0, 1) + (1, 2) - (0, 2), \quad b = (0, 2) + (2, 3) - (0, 3).$$

At which stages are these chains defined and nonzero class in homology? Show that $q = a + b$ at stage 3. Compute the consecutive H_1 maps using $\{[q]\}$ at stage 2, $\{[a], [b]\}$ at stage 3 and $\{[b]\}$ at stage 4.

- (d, ***) Replace the basis at stage 3 by one containing the image of $[q]$. Complete it so that one generator dies at stage 4 and the other survives to stage 4.

- (e,*) Draw the H_0 and H_1 barcodes.